Problems for the Summer School on Symmetries of Combinatorial Structures,
Cuernavaca, July 23 – July 27, 2012
Barry Monson, University of New Brunswick

There are many wonderful resources for the study of symmetric polyhedra in Euclidean space. Try any of [4, 1, 2, 3] or [6] in the references at the end. We have drawn many problems from these sources, without attribution.

Before proceeding, let's review some terminology.

- **Spaces**: Euclidean n-space is $\mathbb{E}^n$, so the plane is $\mathbb{E}^2$, ordinary space is $\mathbb{E}^3$, etc. A unit sphere in $\mathbb{E}^n$ is $\mathbb{S}^{n-1}$.

- **Schläfli symbols** are convenient abbreviations for regular and other polytopes.

  For any integer $p \geqslant 3$, $\{p\}$ will denote a convex regular p-sided polygon (i.e. p-gon). These are all similar for a fixed p, so we can often neglect size.

  The symbols $\{2\}$ and $\{\infty\}$ are also meaningful.

  The regular convex polyhedron $\mathcal{P} = \{p, q\}$ (Platonic solid) likewise has identical p-gonal faces, q around each vertex. If one chops off near a vertex A in the most symmetrical way, one exposes a $\{q\}$, the *vertex-figure* at vertex A.

  So we have $\{3, 3\}$ (tetrahedron), $\{3, 4\}$ (octahedron), $\{4, 3\}$ (cube), $\{3, 5\}$ (icosahedron), $\{5, 3\}$ (dodecahedron).

- The regular star-polyhedra were described by

  Kepler (ca. 1619) $\{\frac{5}{2}, 5\}$ (small stellated dodecahedron),
  $\{\frac{5}{2}, 3\}$ (great stellated dodecahedron)

  Poinsot (ca. 1809) $\{5, \frac{5}{2}\}$ (great dodecahedron),
  $\{3, \frac{5}{2}\}$ (great isosahedron)

- For completeness here are the classical convex regular polytopes of higher rank, their Schläfli symbols and the corresponding symmetry groups (=finite Coxeter groups with string diagram):

name	symbol	# facets	(Coxeter) group	order
$n = 4$:				
simplex	$\{3, 3, 3\}$	5	$A_4 \simeq S_5$	5!
cross-polytope	$\{3, 3, 4\}$	16	B_4	384
cube	$\{4, 3, 3\}$	8	B_4	384
24-cell	$\{3, 4, 3\}$	24	F_4	1152
600-cell	$\{3, 3, 5\}$	600	H_4	14400
120-cell	$\{5, 3, 3\}$	120	H_4	14400
$n > 4$:				
simplex	$\{3, 3, \ldots, 3\}$	$n + 1$	$A_n \simeq S_{n+1}$	$(n + 1)!$
cross-polytope	$\{3, \ldots, 3, 4\}$	2^n	B_n	$2^n \cdot n!$
cube	$\{4, 3, \ldots, 3\}$	$2n$	B_n	$2^n \cdot n!$

- **Euler's Formula** states that for any convex polyhedron with v vertices, e edges and f facets, we have

$$v - e + f = 2 \ . \tag{1}$$

Problems

1. Give Euler's formula for a convex n-dimensional polytope.
2. Prove that no convex polyhedron can have exactly 7 edges. (We aren't assuming anything about symmetry here.)
3. What are the possible edge numbers for a general convex polyhedron in $\mathbb{E}^3$?
4. Show that a general convex polyhedron in $\mathbb{E}^3$ has either a triangular face or a vertex of degree 3 (or both).
5. Show that the edge number e for $\{p,q\}$ is given by
$$e^{-1} = p^{-1} + q^{-1} - \frac{1}{2}.$$
6. Determine the section of (i) a $\{3,3\}$ by a plane midway between two opposite edges; (ii) a $\{4,3\}$ by a plane midway between two opposite vertices; (iii) a $\{5,3\}$ by a plane midway between two opposite vertices.
7. Prove that $\tau := 2\cos(\frac{\pi}{5})$ satisfies $\tau^2 = \tau + 1$.
8. Find a formula for the dihedral angle of $\{p,q\}$. (Clearly this angle is a symmetry invariant; so your answer should be a function of just p and q.)
9. Every regular polyhedron $\{p,q\}$ can be inscribed in a sphere. If the (common!) length of all edges in $\{p,q\}$ is 1, find a formula for the circumradius R.
10. Show that a $\{3,3\}$ can be inscribed in a cube $\{4,3\}$. (The vertices of the regular tetrahedron are found amongst those of the cube.) In fact, how many such tetrahedra can be found in one cube?
11. Show that a $\{4,3\}$ can be inscribed in a dodecahedron $\{5,3\}$. How many such cubes are there in one dodecahedron?
12. Find ordinary rectangular coordinates for the regular isosahedron $\{3,5\}$.
13. Prove that $\{3,3\}$ and $\{3,4\}$ have supplementary dihedral angles.

14. Prove that space $\mathbb{E}^3$ can be tiled in a symmetric and face-to-face way by copies of $\{3,3\}$ and $\{3,4\}$.

15. Give the Schläfli symbol for the usual tiling of space by identical unit cubes. (This regular tessellation is an example of an infinite abstract regular 4-polytope.)

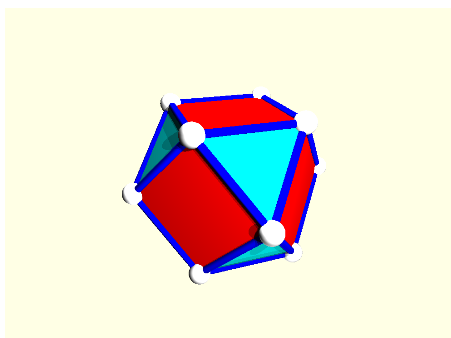
16. Which of the Platonic solids is described by $|x| + |y| + |z| \leqslant 1$ in $\mathbb{R}^3$?

17. Find vertices for the 4-cube $\{4,3,3\}$ and cross-polytope $\{3,3,4\}$ in $\mathbb{E}^4$.

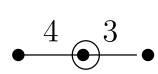
18. Determine v, e, f for the Kepler-Poinsot star-polyhedra $\{\frac{5}{2}, 5\}$, $\{\frac{5}{2}, 3\}$, $\{5, \frac{5}{2}\}$, and $\{3, \frac{5}{2}\}$. What happens with Euler's formula?

19. Which of the star-polyhedra have a face vector $[v, e, f]$ identical to the face vector of some Platonic solid? Is there anything meaningful in this comparison?

20. The cuboctahedron is perhaps the most familiar uniform polyhedron (or Archimedean solid) which is not regular:



Sometimes the cuboctahedron is given the Schläfli symbol $\left\{ \begin{matrix} 3 \\ 4 \end{matrix} \right\}$. But a more informative notation is the decorated diagram



Show how a solid cuboctahedron is situated as a subset of the cube with vertices $(\pm 1, \pm 1, \pm 1)$. What then are the vertices of the cuboctahedron?

21. Take two identical $1 \times 1 \times 1$ cubes. Dissect the second of these into six square based pyramids, with common apex at the centre of the cube. Then place these pyramids on the faces of the first cube. What new polyhedron results? Is it convex? What is its group of symmetries?

    Find the face vector of this object. Compare it with the face vector of the cuboctahedron. What is going on?

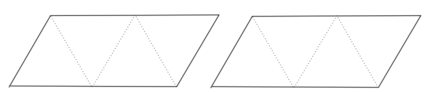
22. The polyhedron in the previous problem is called the *rhombic dodecahedron*. It has two kinds of vertices; however, its faces are all alike, being parallelograms of a certain shape.

    Show that the rhombic dodecahedron is a *space-filler*, that is, congruent copies can be arranged to tile $\mathbb{E}^3$ in a face-to-face manner (indeed in a very symmetric way).

23. Take a rectangular strip of paper, say 2cm $\times$ 20cm and weave it loosely into an ordinary (trefoil or overhand) knot. Now carefully tighten the knot, snug up the strip and flatten it. What regular polygon do you get? Why?

Some Weaving Problems([5, 7])

24. Construct two paper strips of 4 equilateral triangles, as in the following figure. (You could print out the template at the end of these problems.)

    Crease the strips (along the dotted lines, of course). Now weave together the two strips, in over-under fashion, to get a regular tetrahedron $\{3, 3\}$. There should be no loose bits; the ends of the strips should be tucked away, and your model should be sturdy.

    Suppose the strips are coloured. Weave the strips so that every colour has equal exposed area. How many different colour groups can arise? That is, what subgroups of S_4 respect the colouring in a properly constructed model?

25. Make a cube in similar fashion using three strips of five squares (in three distinct colours). What colour groups arise?

    If you use the template at the end, you should be able to inscribe your woven tetrahedron inside your woven cube.

26. See [5] or [7]. Investigate the construction of the remaining Platonic solids, as well as other symmetric polyhedra.

27. Cut out any acute angled triangle and fold up along the joins of the mid-points of the edges. The tetrahedron which results is called a *tetragonal disphenoid*. Prove that the resulting polyhedron has congruent faces and congruent dihedral angles; however, it is usually not regular.

28. Find a tetragonal disphenoid which tiles $\mathbb{E}^3$.

29. Prove that the six edges of any tetrahedron can be realized as the diagonals (one each) of the six faces of a a suitable parallelopiped.

    What sort of tetrahedron gives a rectangular parallelopiped (i.e. brick)?

    What further restriction yields a tetragonal disphenoid (see item (27) above)?

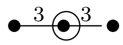
30. **Resist!** Throw away the faces of $\{p, q\}$. What remains is an interesting and symmetric graph called the 1-skeleton of the solid. Suppose a 1Ω resistor is placed along each edge. What is the net resistance between a pair of nodes in the resulting circuit?

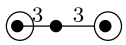
    (There is one such resistance for the tetrahedron, three for the cube, etc.)

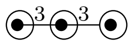
31. You have two identical wooden cubes $\mathcal{A}$ and $\mathcal{B}$. Is it possible to drill an ordinary cylindrical hole through $\mathcal{A}$, along which you can pass $\mathcal{B}$?

Wythoff's Construction for Uniform Polyhedra

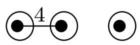
32. Interpret these symbols:

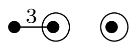
    (a) 

    (b) 

    (c) 

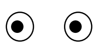
    (d) 

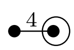
    (e) 

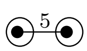
    (f) 

33. In fact, Wythoff's construction remains significant in $\mathbb{E}^n$. Now the dimension n equals the number of nodes in the diagram. So interpret these diagrams:

    (a) 

    (b) 

    (c) 

    (d) 

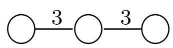
34. Assign diagrams to various other Archimedean solids: truncated tetrahedron, truncated octahedron, truncated cube, cuboctahedron, rhombicuboctahedron, rhombiicosidodecahedron, truncated icosidodecahedron.

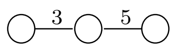
35. In [3, pp. 17–18] Coxeter describes a variant of Wythoff's construction in which we employ just the subgroup $[p, q]^+$ of rotations in the reflection group $[p, q]$.

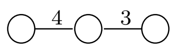
    Recall that the mirrors of reflections in $[p, q]$ cut an invariant sphere into copies of a basic spherical triangle Φ, whose angles are $\pi/p, \pi/q, \pi/2$. These copies correspond in $1 - 1$ fashion to the group elements. In particular, the base triangle Φ, which we will colour white, corresponds to the identity e. The triangles can then be coloured alternately black and white, so that the rotation subgroup $[p, q]^+$ corresponds to the white triangles; the black triangles correspond to the other coset (of opposite symmetries, either reflections or rotatory reflections).

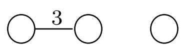
We pick a vertex v in the *white* base triangle Φ. Its convex hull will be a solid which is symmetric under at least $[p, q]^+$. Typically, we can choose v in one way so that all resulting polygonal faces are regular. In this case we may summarize the construction by a diagram with 'empty nodes', as below.

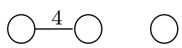
Interpret the diagrams:

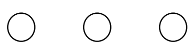
(a) 

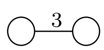
(b) 

(c) 

(d) 

(e) 

(f) 

(g) 

References

[1] H. S. M. Coxeter, *Introduction to Geometry*, John Wiley and Sons, New York, 2nd ed., 1969.

[2] ——, *Regular Polytopes*, Dover, New York, 3rd ed., 1973.

[3] ——, *Regular Complex Polytopes*, Cambridge University Press, Cambridge, UK, 2nd ed., 1991.

[4] H. M. Cundy and A. P. Rollett, *Mathematical Models*, Oxford University Press, Oxford, 2nd ed., 1961.

[5] M. Gardner, *Plaiting polyhedrons*, in Wheels, life and other mathematical amusements, W. H. Freeman, New York, 1983, pp. 106–114.

[6] P. McMullen and E. Schulte, *Abstract Regular Polytopes*, vol. 92 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, UK, 2002.

[7] J. Pedersen, *Platonic solids from strips and clips*, The Australian Mathematics Teacher, 30 (1974), pp. 130–133.

Strips for Weaving a regular tetrahedron {3,3} and a cube {4,3}