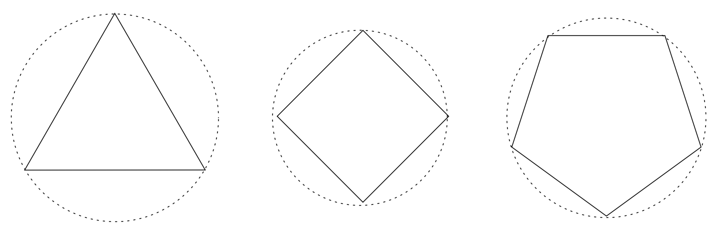
The symmetry groups for polygons, polyhedra, polytopes of the most symmetric kind

1. For an integer $p \geqslant 2$, suppose $v_1, \ldots, v_p$ are equally spaced points on a circle. Connect these in cyclic order by edges $e_j = [v_j, v_{j+1}]$, for $1 \leqslant j \leqslant p$, taking subscripts modulo p. Thus e_p closes the cycle by connecting v_p to v_1.

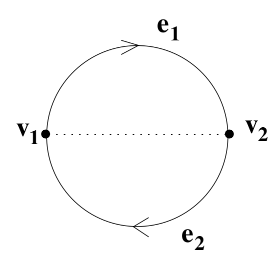
   We obtain a *regular p-gon*, denoted $\{p\}$:



{3} {4} {5}

But what if p = 2?

{2}



2. Suppose $p \geqslant 3$. We see that $\{p\}$ is a familiar convex polygon. After replacing each edge by the circular arc it spans, we obtain *circular p-gons* with the same abstract structure. (Think of the vertices and edges as defining a *graph*.)

   It therefore makes sense to say that the *digon* $\{2\}$ (see the figure above) has two vertices and two edges. We just cannot separate the edges if we insist on using straight line segments.

3. What is the symmetry group $\mathbb{D}_p$ of the regular polygon $\{p\}$?

   The mirrors for the various reflection symmetries are lines, all passing through the centre O of the circumscribing circle. These mirrors divide the plane into angular regions. The number of such regions will equal the order of the symmetry group.

Definition 0.1. *For $p \geqslant 2$, the symmetry group of a regular p-gon $\{p\}$ is denoted $\mathbb{D}_p$.*

Warning. There is much disagreement about whether the subscript p should be a little different (see below).

4. We have observed for the equilateral triangle and the square that we get generating reflections by taking two distinct mirrors separated by the smallest possible angle. For a $\{p\}$, this angle will be π/p.

   To make the process a little more susceptible to generalization in higher dimensions, let us do the following. A *flag* of the polygon is a pair consisting of an incident vertex and edge. (It will turn out that the order of the symmetry group $\mathbb{D}_p$ equals the number of such flags. Of course, this order equals $2p$, since every vertex lies on 2 edges. Some authors, including at other times myself, use the order $2p$ as subscript, instead of just p itself.)

   So choose any one flag as your *base flag*, say $[v_1, e_1]$ to be specific. Let

   - r_0 be the reflection in the perpendicular bisector of edge e_1. Thus

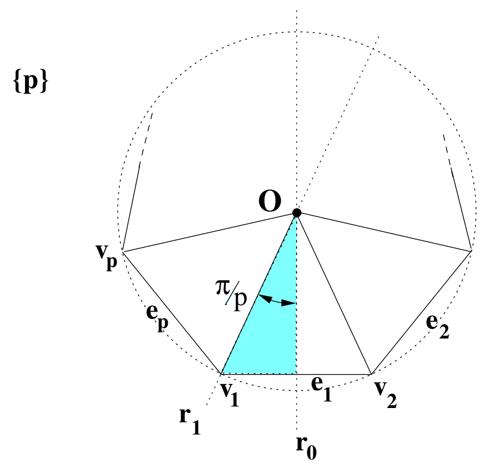
     $$\begin{aligned} r_0 \quad : \quad & v_1 \to v_2 \\ & e_1 \to e_1 \end{aligned}$$

     In short, r_0 moves the 0-dimensional 'face' (= vertex) of the base flag and fixes the 1-dimensional face (= edge). (As one entity, edge e_1 is fixed; of course, it is flipped end-for-end in the process.)

   - r_1 be the reflection in the line joining O to v_1. Thus

     $$\begin{aligned} r_1 \quad : \quad & v_1 \to v_1 \\ & e_1 \to e_p \end{aligned}$$

     In short, r_1 moves the 1-dimensional 'face' (= edge) of the base flag and fixes the 0-dimensional face (= vertex).



A *fundamental region* for the action of the symmetry group $\mathbb{D}_p$ on the polygon $\{p\}$ has been shaded in. Repeated application of r_0 or r_1 will move this region to all $2p$ available positions.

Thus $\mathbb{D}_p$ has generators r_0, r_1; once again we see that the order is $2p$.

The group $\mathbb{D}_p$ is called *dihedral* since we imagine it generated by two reflections. In fact, we can make an actual *kaleidoscope* corresponding to this group by using two real mirrors, hinged at the angle π/p and placed vertically on a table.

5. A presentation for $\mathbb{D}_p$.

   We can reason as we did for the equilateral triangle or square: explicitly list the $2p$ elements of the group. Alternatively one can use coset enumeration on the trivial subgroup. In any case, a presentation is

$$\mathbb{D}_p : \ \langle \, r_0, r_1 \mid r_0^2 = r_1^2 = (r_0 r_1)^p = 1 \rangle$$

   This sort of presentation means that $\mathbb{D}_p$ is an example of a *Coxeter group*. These groups appear all throughout mathematics, often in places which wouldn't seem to have much to do with polygons or polyhedra.

   Very Important Convention. I like to compose geometric and algebraic mappings *left-to-right*. Thus

$$r_0 r_1 \text{ means first apply } r_0 \text{ then apply } r_1.$$

6. Exercises.

(a) Fix a background label 'j' next to vertex v_j for each j. Use this to represent the generators r_0 and r_1 as permutations R_0, R_1 on $\{1, \ldots, p\}$.

Compute $R_0 R_1$ in this representation. For which p is the resulting group $\langle R_0, R_1 \rangle$ of permutations isomorphic to $\mathbb{D}_p$? In other words, can the permutations ever fail us?

(b) **Pedantic Note**: r_j is a geometric reflection, whereas R_j is a permutation, a quite different beast. Thus, if we are trying to be very careful, we need such notation. But in practice, out of the sight of fussy professors, we might conveniently confuse r_j and R_j.

(c) Clearly $\langle R_0, R_1 \rangle$ is a subgroup of $\mathbb{S}_p$. Can the subgroup equal the whole group for any p?

(d) What goes wrong with the above permutations when $p = 2$? Correct that to give a *faithful* permutation representation of $\mathbb{D}_p$, say as a subgroup of $\mathbb{S}_4$.

(e) Compute on geometrical grounds the number of conjugacy classes in $\mathbb{D}_p$. (Intuitively, a conjugacy class consists of all symmetries which act in the 'same' geometrical way on the p-gon. Here 'same' will mean 'up to relocation via any and all elements of the surrounding group, here $\mathbb{D}_p$.)

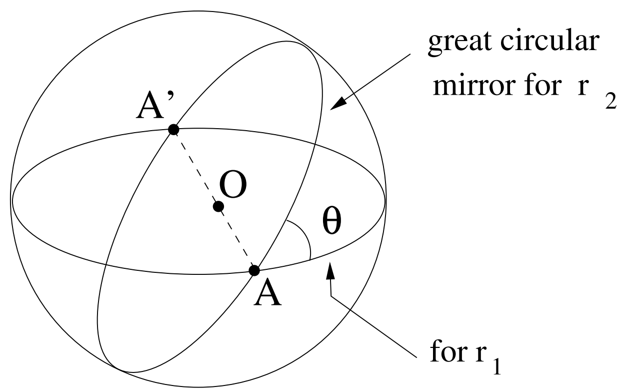
Find out how to retrieve a permutation version of $\mathbb{D}_p$ in GAP and check your conjectures about the number of conjugacy classes for several small values of p.

The Finite reflection groups G

Our examples indicate that regular polygons and polyhedra, and presumably their higher dimensional kin, have symmetry groups generated by reflections. We survey these groups in ordinary space.

7. Suppose then that G is a *finite group generated by reflections* in $\mathbb{R}^3$ and pick any point P. Its orbit, $\mathrm{Orb}_G(P)$, is finite, since G is finite. The orbit therefore has a well-defined centroid O.

   But since G consists of isometries, each of which rearranges the points in the orbit, it must be that O is *fixed by every element of the group*. Furthermore, since G consists of isometries, this means that G fixes as an entity any sphere centred at O. In short, we can track the action of G by examining how it acts on the unit sphere $\mathbb{S}^2$ centred at O:

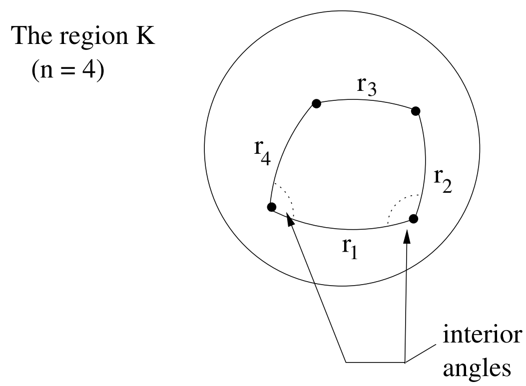


   Any reflection in G has a plane mirror passing through O. This mirror meets $\mathbb{S}^2$ in a *great circle*. Two such mirrors intersect in a line which meets the sphere at *antipodal* points A and A', as shown above. Notice that two distinct great circles always intersect in a pair of antipodal points.

   The (dihedral) angle θ from one such mirror to another appears to a spherical bug living on $\mathbb{S}^2$ as an angle on the surface of the sphere. Just as in the Euclidean plane, we find that a product $r_1 r_2$ of reflections equals a rotation through angle 2θ with centre A. If the angle appears to be anticlockwise as we view A from outside the sphere, then it will appear clockwise at A'.

8. Now experience with kaleidoscopes informs us that all the mirrors for reflections in G will cut $\mathbb{S}^2$ into various congruent, spherical polygons. Any one of these polygons (call it K) serves as a *fundamental region* for the action of the group G. This means that by repeated reflection in the great circles bounding K, we are able to cover the entire sphere once over.

   From another point of view, K is a smallest region *enclosed* by mirrors but not penetrated by any mirror of symmetry. If the polygon K has n sides, let's label the bounding reflections $r_1, \ldots, r_n$ is cyclic order:



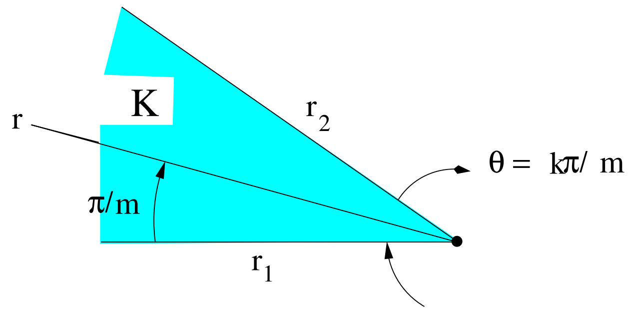
9. Look at the reflections r_1, r_2 in two consective sides of K. Suppose the angle (interior to K) from side 1 to side 2 is θ. Then $r_1 r_2$ is a rotation with angle 2θ. But $r_1 r_2$ has some period $m \geqslant 2$ as an element of the group G, so $(r_1 r_2)^m = 1$.[1] (If $m = 1$ we would have $r_1 = r_2$ and the two mirrors would coincide.) Since the identity 1 is a rotation through some multiple of 2π, we must have $m(2\theta) = 2\pi k$ for some integer k. Being a period, m is minimal for this last condition, so m and k are relatively prime. Thus $1 = \gcd(m, k) = xk - ym$, for certain $x, y \in \mathbb{Z}$. Now we see that the rotation $g := (r_1 r_2)^x$ in G has angle

$$x(2\theta) = \frac{x 2\pi k}{m} = \frac{2\pi}{m} + 2\pi y \; .$$

   Thus rotation g has angle $2\pi/m$.

[1] Note that the symbol 1 is used in various ways: an integer, the identity in a group generated by relections, etc. This abuse of language is most convenient.

Now let r be reflection in the line x located π/m along from the mirror for r_1:



This line is interior to the region K if $k > 1$; but we don't yet know if r is in the group. However, $r_1 r$ is also a rotation through $2\pi/m$, so $r_1 r = g$ and $r = r_1 g$ really is in the group G. This contradicts our construction of the region K if $k > 1$. (We chose K so that no mirrors penetrate its interior.)

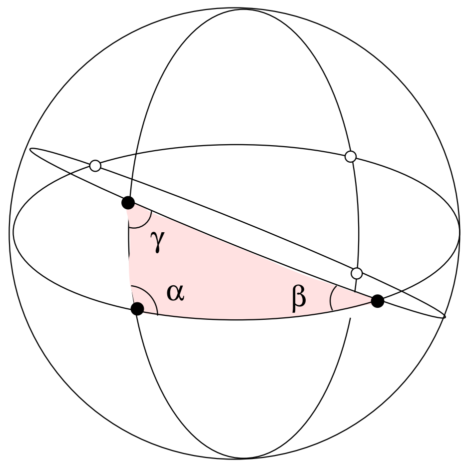
10. **Conclusion**. Each interior angle of K has the form π/m for some integer $m \geqslant 2$. (Such angles are called *submultiples* of π).

11. **The area of a spherical polygon**

    A *lune* is a region of $\mathbb{S}^2$ bounded by two great semicircles. In the figure on page 5 you can see four lunes terminating at A and A'. Look at the lune specified by the angle θ. The symmetry of the sphere clearly implies that this area is directly proportional to θ. Since the whole unit sphere has area 4π we conclude that

    the area of a lune with polar angle θ is 2θ.

12. Let's look now at a spherical triangle with angles α, β, γ:



Extending its sides we obtain various lunes which intersect in a congruent antipodal triangle (indicated by 'open' vertices). Let L be the area of the triangle. Observing how the various lunes cover the sphere, we get

$$4\pi = 2(2\alpha + 2\beta + 2\gamma) - 4L,$$

so that

the area of a spherical triangle with angles α, β, γ is
$$\alpha + \beta + \gamma - \pi,$$
(the angular excess).

13. Let's return to our fundamental region K, which is a spherical n-gon ($n \geqslant 2$) with angles of the form $\frac{\pi}{p_1}, \frac{\pi}{p_2}, \ldots, \frac{\pi}{p_n}$, where we have seen each integer $p_j \geqslant 2$. Now subdivide K into $n-2$ spherical triangles and employ the angular excess. We conclude that

$$K \text{ has area } \pi[\tfrac{1}{p_1} + \cdots + \tfrac{1}{p_n} - (n-2)].$$

But this area is positive. On the other hand, each $\frac{1}{p_j} \leqslant \frac{1}{2}$, so that
$$0 < \frac{n}{2} - (n-2)\ .$$

We conclude that $n = 2$ or 3. This immediately leads to an easy enumeration of cases, as well as a formula for the orders of the resulting reflection groups. We may include the degenerate case $n = 1$ for completeness.

14.

Theorem 0.2. *Let G be a finite reflection group in ordinary Euclidean space. Then G belongs to one of the following classes:*

(a) *$G = \langle r_1 \rangle$ is generated by one reflection and has order 2. In this case K is a hemisphere.*

(b) *$G = \langle r_1, r_2 \rangle$ is a dihedral group $\mathbb{D}_p$ for some $p \geqslant 2$. Here G has order $2p$ and K is a lune bounded by semicircles with polar angle π/p.*

(c) *$G = \langle r_1, r_2, r_3 \rangle$ is generated by three reflections whose mirrors bound a spherical triangle K. The actual cases are*

- *$(p_1, p_2, p_3) = (2, 2, p)$ for any integer $p \geqslant 2$. Here G has order $4p$ and can serve as the symmetry group of of a uniform p-gonal right prism.*
- *$(p_1, p_2, p_3) = (2, 3, 3)$. Here G has order 24, is isomorphic to the symmetric group $\mathbb{S}_4$ and serves as the symmetry group of the regular tetrahedron $\{3, 3\}$.*
- *$(p_1, p_2, p_3) = (2, 3, 4)$. Here G has order 48 and can serve as the symmetry group of the cube $\{4, 3\}$ or regular octahedron $\{3, 4\}$. (Here $G \simeq \mathbb{S}_4 \times \mathbb{C}_2$.)*
- *$(p_1, p_2, p_3) = (2, 3, 5)$. Here G has order 120 and can serve as the symmetry group of the regular dodecahedron $\{5, 3\}$ or regular icosahedron $\{3, 5\}$. (G is not isomorphic to $\mathbb{S}_5$; instead $G \simeq \mathbb{A}_5 \times \mathbb{C}_2$.)*

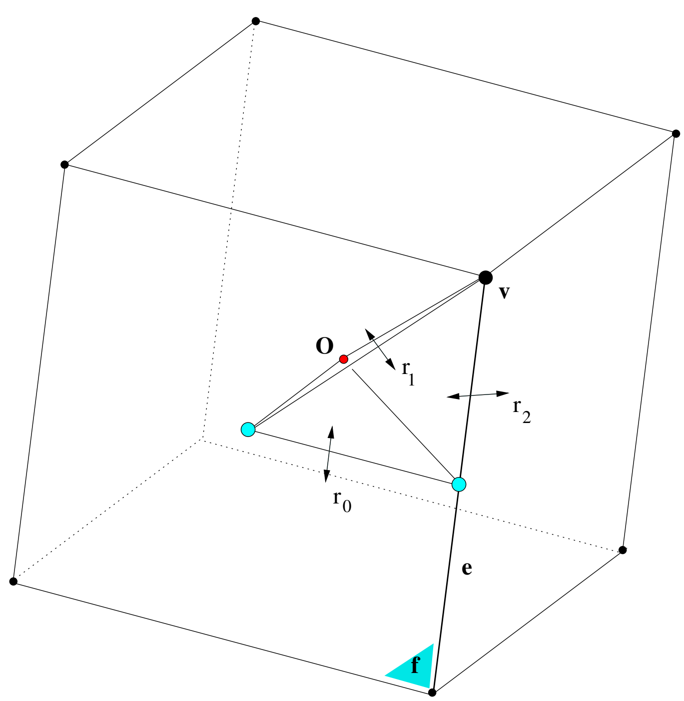
We note that the order of the symmetry group $[p, q]$ for the regular polyhedron $\{p, q\}$ is

$$\frac{4}{\frac{1}{p} + \frac{1}{q} - \frac{1}{2}} = \frac{8pq}{4 - (p-2)(q-2)} \; .$$

The Abstract Cube

1. The relation between the cube $\mathcal{P}$ and its symmetry group G is quite typical of what happens for general regular polyhedra (regular polytopes of rank $n = 3$). The extension to regular polytopes of higher rank ($n \geqslant 4$) or even to lower ranks (eg. polygons, of rank $n = 2$) is quite natural. Therefore, instead of proving general things, we will just use the cube to suggest a believable description of the basic theory of abstract regular polytopes.

2. The cube has *Schläfli symbol* $\{4, 3\}$ (after Ludwig Schläfli, a 19th century Swiss geometer). Here the '4' indicates that the faces of $\mathcal{P}$ are squares (Schläfli symbol $\{4\}$); the 3 indicates that 3 squares surround each vertex. More precisely, the vertex-figure of a typical vertex like v below is the equilateral triangle $\{3\}$ formed by the three vertices adjacent to v. (Sketch it in yourself.)



3. The group $G = G(\mathcal{P})$ is known to have presentation

   $$G = \langle r_0, r_1, r_2 \,|\, r_0^2 = r_1^2 = r_2^2 = (r_0 r_1)^4 = (r_1 r_2)^3 = (r_0 r_2)^2 = 1 \rangle \; .$$

   (You could check the order 48 by coset enumeration. In the 1930's Coxeter used geometric arguments to solidify our understanding of these kinds of groups in all dimensions.) Notice where the 4 and 3 appear. And recall that the period 2 means that r_0 commutes with r_2.

   A *Coxeter group* is a group presented in this way as having generators r_j of period 2 subject only to those further relations which specify the periods of products of two distinct generators.

   Thus, if there are n generators $r_0, r_1, \ldots, r_{n-1}$, we will have at most $\begin{pmatrix} n \\ 2 \end{pmatrix}$ further defining relations.

   Some such relation could be missing. This is an admission that we allow the period to be ∞. For example, the rank 2 Coxeter group

   $$\langle r_0, r_1 \,|\, r_0^2 = r_1^2 = 1 \rangle$$

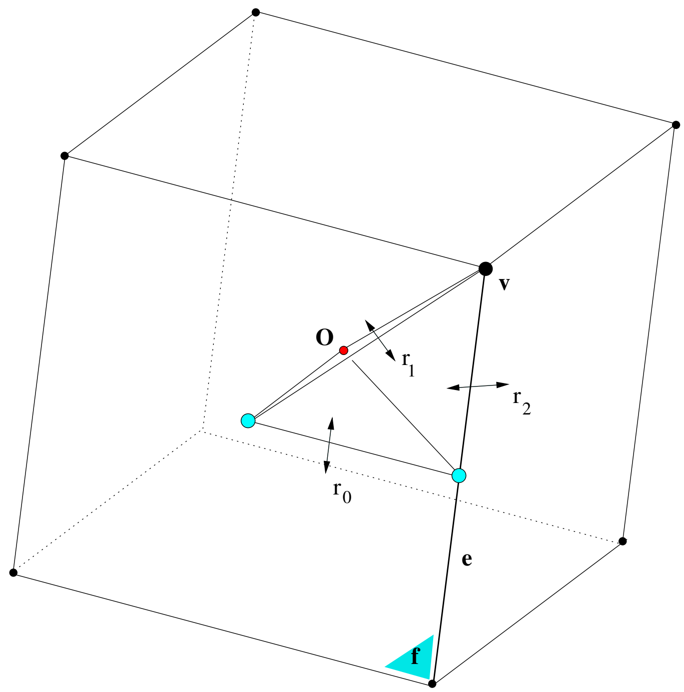
   actually is infinite. The generators can be interpreted as reflections in distinct parallel lines in the plane. This gives the symmetry group of the infinite regular polygon $\{\infty\}$ whose vertices are all points with integer coordinates on a line perpendicular to the mirrors. Sketch this yourself, taking care to place the mirrors for r_0, r_1 correctly.

4. The presentation for any Coxeter group can be encoded in a most useful *Coxeter diagram*. The diagram for G above is

   $$\bullet \overset{4}{\text{———}} \bullet \overset{3}{\text{———}} \bullet$$

   The three nodes correspond left-to-right to r_0, r_1, r_2. You can see how the 'rotational' periods are indicated by the branch labels. Crucially, non-adjacent nodes correspond to commuting reflections. This is a very useful trick.

5. Let's return to the picture of the cube:



Now we want to understand how to extract the generating reflections from the geometrical set-up. The first step is to chooses a *base flag*, i.e. an incident [vertex, edge, square face] triple. The flag $[v, e, f]$ is indicated. We may identify each of these components by its centroid. As a result, we get the isosceles right triangle whose vertices are v, the midpoint of e and the centre of f. This triangle does look a bit like a flying pennant.

There are $48 = 8 \cdot 6$ copies of the pennant on the surface of the cube. We see once more why G has order 48.

If you now join these three points to the body centre O for the cube itself, then you get the framework for an actual *trihedral kaleidoscope*.

The generating reflections now arise in a systematic way:

- reflection r_0 moves only the dim 0 component of the base flag (move v, globally fix e, f as entities);
- reflection r_1 moves only the dim 1 component of the base flag (move e, globally fix v, f as entities);
- reflection r_2 moves only the dim 2 component of the base flag (move f, globally fix v, e as entities).

The base flag is moved by r_j to the so-called $j - adjacent$ flag. Take a moment to find these three flags and shade in their pennants.

Now extract the abstract...

6. In order to count ingredients of each rank, we might employ stabilizers. Observe that

   - the stabilizer of the rank 0 element in the base flag is
   $$G_0 = \mathrm{Stab}_G(v) = \langle r_1, r_2 \rangle$$
   - the stabilizer of the rank 1 element in the base flag is
   $$G_1 = \mathrm{Stab}_G(e) = \langle r_0, r_2 \rangle$$
   - the stabilizer of the rank 2 element in the base flag is
   $$G_2 = \mathrm{Stab}_G(f) = \langle r_0, r_1 \rangle$$

   In brief, the stabilizer of the rank j element in the base flag is
   $$G_j = \langle r_i \,:\, i \neq j, 0 \leqslant i \leqslant n-1 \rangle,$$
   where $n = 3$ for the cube, of course. The same description works for regular polytopes of any rank.

   In any regular polytope $\mathcal{P}$, the symmetry group $G = G(\mathcal{P})$ will be transitive of 'faces' of each particular rank j: there is just one orbit for each. Thus the number of j-faces in $\mathcal{P}$ equals
   $$\frac{|G|}{|G_j|}.$$

7. Now recall how one proves this. We exhibit a $1-1$ correspondence between j-faces and right cosets of the stabilizer in G. If x denotes the j-face in the base flag, then as g runs through G, we have

$$x^g \leftrightarrow G_j g$$

From an abstract point of view: j-faces <u>are</u> the cosets $G_j g$.

8. We have the *ingredients* of $\mathcal{P}$. What about *assembly instructions*? In other words, can we use the cosets to say when some j-face 'lies on' or 'is incident with' some k-face?

   Again we look at the cube to see what must happen. For instance, when does a vertex (face of rank $k=0$) lie on a square (face of rank $j=2$)?

   Well, a typical vertex is v^g and a typical square face is f^h, where $g, h \in G$. Note that g and h might well be different, but due to transititity, this does cover all cases.

   Working in one direction, let us assume that vertex v^g lies on square f^h. Since h^{-1} is a symmetry in G, this means

$$(v^g)^{h^{-1}} = v^{(gh^{-1})} \text{ lies on the base square } f = f^1 = f^{(hh^{-1})}.$$

   Now we appeal in an inductive way to our knowledge of lower-rank objects, in this case, the square f whose own symmetry group is isomorphic to $G_2 = \langle r_0, r_1 \rangle$. There must be some $y \in G_2$ such that $v^{(gh^{-1})} = v^y$. This in turn implies that $gh^{-1}y^{-1}$ fixes v, so that $gh^{-1}y^{-1} \in G_0 = \langle r_1, r_2 \rangle$. Thus

$$\begin{aligned} G_0(gh^{-1}y^{-1}) &= G_0 \\ G_0 g &= G_0(yh) \end{aligned}$$

   At the same time, since $y \in G_2$, we have $G_2 y = G_2$, so that $G_2 h = G_2(yh)$. This shows that $G_0 g$ and $G_2 h$ have a common representative yh:

$$yh \in G_0 g \cap G_2 h\ .$$

The converse implication is easier. If we assume that $G_k g \cap G_j h$ is non-empty, then say $z \in G_k g \cap G_j h$. In other words, $z = xg$, where x fixes v and $z = yh$, where y fixes f.

Back in the base flag v is incident with f, so that if we apply z we conclude that vertex $v^z = v^{(xg)} = v^g$ is incident with square $f^z = f^{(yh)} = f^h$.

From an abstract point of view: If $k \leqslant j$, the k-face $G_k g$ is incident with the j-face $G_j h$ if and only if

$$G_k g \cap G_j h \neq \emptyset \ .$$

9. We have seen how to 'reconstruct' the cube, at least in combinatorial essentials, purely from the point of view of its symmetry group G.

   Much the same sort of thing is possible for any abstract regular n-polytope, so that many questions concerning polytopes can be reconfigured as questions concerning a suitable group G.

   What sort of groups are suitable? Schulte proved in the early 1980's that the regular n-polytopes correspond in a precise way to *string C-groups G*. Such a group has these properties:

   - it is generated by n specified elements $r_0, \ldots, r_{n-1}$ each of period 2. (The subscripts emphasize that the ordering of these generators is important.)
   - these elements satisfy certain relations
     - $(r_k r_j)^2 = 1$, if $k < j - 1$ (indicating commuting generators).
     - $(r_{j-1} r_j)^{p_j} = 1$, for $1 \leqslant j \leqslant n - 1$ (think 'rotational periods'). Here each $p_j \in \{2, 3, 4, \ldots, \infty\}$.

     Remarks. We may well need other relations of a type not listed above to effect a presentation. But <u>if</u> no other relations are needed, then the special sort of group that results is called a *Coxeter group with string diagram.* The diagram is a simple left to right string of nodes. If $p_j > 2$, then there is branch labelled p_j connecting the $(j - 1)$st node to the jth node.

     The Schläfli symbol for the group (and polytope) is $\{p_1, \ldots, p_{n-1}\}$.

     We are not yet done with special properties for G...

The group must also satisfy an

- *intersection condition*: for any subsets $I, J \subseteq \{0, 1, \ldots, n-1\}$, we have

$$\langle r_k : k \in I \rangle \cap \langle r_k : k \in J \rangle = \langle r_k : k \in I \cap J \rangle \ .$$

Example. Many of these intersections hold for trivial reasons. In the case of an abstract regular polyhedron (rank $n = 3$), the crucial condition to be checked is

$$\langle r_0, r_1 \rangle \cap \langle r_1, r_2 \rangle = \langle r_1 \rangle \ .$$

The direction $\supseteq$ holds for sure. So what we really have to show is $\subseteq$, namely

$$g \in \langle r_0, r_1 \rangle \cap \langle r_1, r_2 \rangle \Rightarrow g \in \langle r_1 \rangle \ .$$

Remark. It turns out that Coxeter groups do satisfy the intersection condition. This is tricky to prove. In other words, it is the unstated 'extra' relations that can befoul the intersection condition. Much of one's time in regular polytope theory is spent dealing with this fact.