Euclidean Space and its Isometry Group

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A. **Introduction and References**.

There are numerous sources which cover the material below. Many of the problems concerning regular polygons and polyhedra are substantially adapted from the beautiful treatment in

[1] H. S. M Coxeter, Regular Complex Polytopes, Cambridge University Press, Cambridge, 1991, chapters 1–3.

For a different development of the isometry groups and regular polyhedra in various spaces, consult

[2] L. Fejes-Toth, Regular Figures, Pergammon, New York, 1964.

For an accessible and wide ranging treatment of spaces with constant curvature and their groups see

[3] J. Ratcliffe, Foundations of Hyperbolic Manifolds, Graduate Texts in Mathematics, 91, Springer-Verlag, New York, 1994.

We shall mostly work in d-dimensional Euclidean space $\mathbb{E}^d$, though sometimes we need spherical space $\mathbb{S}^d$ (the unit sphere in $\mathbb{E}^{d+1}$), or even hyperbolic space $\mathbb{H}^d$. Although we shall often use purely geometric arguments, particularly in the plane $\mathbb{E}^2$ or in ordinary space $\mathbb{E}^3$, it is still useful to review some analytic methods below.

B. **Euclidean Spaces**

We may define $\mathbb{E}^d$ as a d-dimensional real vector space, equipped with an **inner product** $x \cdot y$ (i.e. positive definite symmetric bilinear form). The **norm** of $x \in \mathbb{E}^d$ is $\|x\| = (x \cdot x)^{1/2}$.

1. Verify the Cauchy-Schwartz inequality for $x, y \in \mathbb{E}^d$:

$$| \, x \cdot y \, | \leq \; \|x\| \; \|y\| \; .$$

   Characterize the case of equality.

2. Show that $(\mathbb{E}^d, \| \cdot \|)$ is a **normed linear space**.

   We thus have a **metric** on $\mathbb{E}^d$: the **distance** from x to y is $\|x - y\|$.

3. Verify that $\mathbb{E}^d$ is then a metric space.

4. Check that $\mathbb{R}^d$, the space of $1 \times d$ row vectors, equipped with the standard inner product $x \cdot y := xy^t$ is a Euclidean d-space.

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Occasionally it will do to consider $\mathbb{E}^d$ as a **linear space** with origin o, say. At other times, we require the natural **affine** structure on $\mathbb{E}^d$. If $x, y \in \mathbb{E}^d$, the (closed) **line segment** joining x to y is

$$[x, y] = \{(1 - \lambda)x + \lambda y : \lambda \in \mathbb{R}, 0 \leq \lambda \leq 1\}$$

Likewise, the **line** joining x and y is

$$\overleftrightarrow{xy} = \{(1 - \lambda)x + \lambda y : \; \lambda \in \mathbb{R}\} \; .$$

5. For $u \in \mathbb{E}^d$, $u \in [x, y]$ if-f

$$\|x - u\| \; + \; \|u - y\| = \|x - y\| \; .$$

We allow $x = y$, in which case the segment or 'line' actually degenerates to a single point.

If $x_1, ..., x_r \in \mathbb{E}^d$, $\lambda_1, ...\lambda_r \in \mathbb{R}$, then

$$x = \lambda_1 x_1 + ... + \lambda_r x_r$$

is

- a linear combination of $x_1, ..., x_r$ in any case;

- an affine combination, if also $\lambda_1 + ... + \lambda_r = 1$
- a convex combination, if furthermore all $\lambda_j \geq 0$.

A subset $U \subseteq \mathbb{E}^d$ is a **linear** (resp **affine**) **subspace** if it is closed under arbitrary linear (resp. affine) combinations of its elements. Similarly, U is a **convex** subset of $\mathbb{E}^d$ if closed under arbitrary convex combinations.

If $U, W \subseteq \mathbb{E}^d$ and $\lambda \in \mathbb{R}$, then we define

$$U + W := \{u + w : u \in U, w \in W\}$$
$$U - W := \{u - w : u \in U, w \in W\}$$
$$\lambda U = \{\lambda u : u \in U\} \ .$$

In particular, if $a \in \mathbb{E}^d$ we let

$$U + a := U + \{a\} \ .$$

6. For $U, W, Z \subseteq \mathbb{E}^d$, $\lambda, \eta \in \mathbb{R}$

$$\begin{aligned} (U + W) + Z &= U + (W + Z) \ , \\ \lambda(U + W) &= \lambda U + \lambda W \ . \end{aligned}$$

When does $(\lambda + \eta)U = \lambda U + \eta U$?

7. Show that U is a linear subspace of $\mathbb{E}^d$

if-f U is closed under the linear operations induced from $\mathbb{E}^d$.

if-f $U + \lambda U \subseteq U$, $\forall \lambda \in \mathbb{R}$.

8. Show that U is an affine subspace of $\mathbb{E}^d$
   if-f $\overleftrightarrow{xy} \subseteq U$ for all $x, y \in U$ (i.e. U is 'line closed')
   if-f U is the translate $L + a$ of a linear subspace $L \subseteq \mathbb{E}^d$.

9. Consider the affine subspace $U \subseteq \mathbb{E}^d$, and suppose $U = L + a$ for *linear* subspace L.

   - Show that L is uniquely determined by U.
   - Show that $L = U - U$.
   - If $b \in U$, show that $U = L + b$.
   - Show that two translates of L are either identical or disjoint.

We say naturally enough that two translates of L are **parallel**. More generally, affine spaces U, W are parallel if one of the **direction spaces** $U - U$, $W - W$ is a subspace of the other.

10. Suppose U is an affine subspace and $b \in U$. For $\lambda \in \mathbb{R}$, define

$$\lambda * U := \text{unique translate of } U \text{ passing through } \lambda b \, .$$

Show that this definition is independent of b and that

$$\lambda * U = \lambda b + (U - U) = (\lambda - 1)b + U \, .$$

Show that $\lambda * U = \lambda U$ if $\lambda \neq 0$. However,

$$0 * U = U - U \, .$$

(P. McMullen has used this useful convention in describing realizations of apeirohedra.)

11. A subset $U \subseteq \mathbb{E}^d$ is convex if-f it contains the line segment $[x, y]$ for all $x, y \in U$.

12. The intersection of any family of linear subspaces is a linear subspace; ditto for affine subspaces, convex subsets.

For any subset $X \subseteq \mathbb{E}^d$, we may define **linear hull** $\text{lin}X$ to be the intersection of all linear subspaces containing X. The **affine hull** $\text{aff } X$ and **convex hull** $\text{conv } X$ are similarly defined.

13. For any X, $\text{lin}X$ is a linear subspace; $\text{aff}\, X$ is an affine subspace; $\text{conv}X$ is a convex set.

14. Prove that $\text{lin}X$ equals the set of all linear combinations of elements of X. Prove analogous statements for $\text{aff}X$ and $\text{conv}X$.

15. Prove

$$\begin{aligned} \text{lin}(\text{lin}X) &= \text{lin}X \\ \text{aff}(\text{aff}X) &= \text{aff}X \\ \text{conv}(\text{conv}X) &= \text{conv}X \\ \text{aff}(\text{conv}X) &= \text{aff}X \end{aligned}$$

The convex hull of a finite set X in $\mathbb{E}^d$ is called a **convex polytope**.

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We now discuss dimension. A subset $X \subseteq \mathbb{E}^d$ is **linearly dependent** if there exist $x_1, ..., x_r \in X$ and $\lambda_1, ..., \lambda_r \in \mathbb{R}$, not all 0, such that

$$o = \lambda_1 x_1 + ... + \lambda x_r \, .$$

If also $\lambda_1 + ... + \lambda_r = 0$, we say X is **affinely dependent**. If this does not happen, then the set X is linearly (resp. affinely) **independent**.

16. X is affinely dependent

    - if-f $X + a$ is affinely dependent, for any $a \in \mathbb{E}^d$.
    - if-f some $x \in X$ is an affine combination of other $x_j \in X$.

17. Let $x_j = (\xi_{j1}, ...\xi_{jd})$, for $0 \leq j \leq r$.
    Then $\{x_0, ..., x_r\}$ is affinely dependent

    - if-f $\{x_1 - x_0, ..., x_r - x_0\}$ is linearly dependent.
    - if-f the matrix

    $$\begin{bmatrix} 1 & \xi_{01} & ... & \xi_{0d} \\ 1 & \xi_{11} & ... & \xi_{1d} \\ \vdots & & & \\ 1 & \xi_{r1} & ... & \xi_{rd} \end{bmatrix}$$

    has rank less than $r + 1$.

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Now let U be any affine subspace of $\mathbb{E}^d$ and let L $(= U - U)$ be its (unique) linear direction space. Thus $U = L + a$, for any $a \in U$. Let $r = \dim(L)$.

18. Every affinely independent subset of U is contained in some maximal affinely independent subset B . Each such B has $r + 1$ elements and $U = \text{aff}(B)$; furthermore, every $x \in U$ is uniquely expressible as an affine combination of elements of B .

    Accordingly, the set $B = \{b_0, b_1, ..., b_r\}$ in the previous problem is called an **affine basis** for U , and we say that $\dim(U) = r$.

19. Interpret this in the case that $\{b_0, b_1, b_2\}$ are the vertices of a triangle in the plane $\mathbb{E}^2$ (here U equals all of $\mathbb{E}^2$). Show that each $x \in \mathbb{E}^2$ can be uniquely expressed as

    $$x = \lambda_0 b_0 + \lambda_1 b_1 + \lambda_2 b_2 \ ,$$

    where $\lambda_j \in \mathbb{R}$ and $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

    Find a geometric meaning for the affine coordinates λ_j . Describe each vertex, edge and supporting line of the triangle using these **affine coordinates**. The three supporting lines divide the plane into 7 regions. Characterize each in terms of affine coordinates.

20. Show that $\{x_0, ..., x_r\}$ is an affine basis for $U = L + a$ if-f $\{x_1 - x_0, ..., x_r - x_0\}$ is a (linear) basis for $L = U - U$.

21. If $U = \text{aff}(X)$, we can choose an affine basis for U from X.

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Let L be any r-dimensional linear subspace of $\mathbb{E}^d$. Recall that we can extend any basis for L to one for $\mathbb{E}^d$, and further use the Gram-Schmidt process to produce an orthonormal basis $\{e_1, ..., e_d\}$ for $\mathbb{E}^d$, where $\{e_1, ...e_r\}$ is an orthonormal basis for L. The **orthogonal complement** of L is the $(d-r)-$dimensional linear space $L^\perp$ spanned by $\{e_{r+1}, ..., e_d\}$. Thus $\mathbb{E}^d = L \oplus L^\perp$.

22. Show that

$$L^\perp = \{x \in \mathbb{E}^d : \; x \cdot u = 0, \; \forall u \in L\} \; .$$

(Thus $L^\perp$ can be defined independently of the basis used above.)

23. Suppose L, M are linear subspaces of $\mathbb{E}^d$. Show that

    (a) $L \subseteq M \Longrightarrow M^\perp \subseteq L^\perp$.

    (b) $(L^\perp)^\perp = L$

    (c) $L + M$ and $L \cap M$ are linear subspaces of $\mathbb{E}^d$.

    (d) $(L + M)^\perp = L^\perp \cap M^\perp$.

    (e) $(L \cap M)^\perp = L^\perp + M^\perp$.

24. (a) Give a sensible definition for an orthogonal complement to an affine subspace L.

    (b) Given any affine subspace L and point $x \in \mathbb{E}^d$, show that there exists a unique affine subspace M through x and orthogonal to L.

25. Let L be a subset of $\mathbb{E}^d$.

    (a) Show that L is a 0-dimensional affine subspace of $\mathbb{E}^d$ if-f $L = \{x\}$ for some $x \in \mathbb{E}^d$. (We often simply write $L = x$.)

    (b) Show that L is a 1-dimensional affine subspace if-f $L = \overleftrightarrow{xy}$ (a line, with $x \neq y$).

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A $(d-1)-$dimensional (affine) subspace H is called a **hyperplane**.

26. Show that any hyperplane H is defined by a linear equation

$$x \cdot n = k$$

for some non-zero **normal** vector n. Indeed, show that any line of the form $L = \mathbb{R}n + c$ is completely orthogonal to H.

27. Compute the (shortest) distance from the point y to (a point on) the hyperplane $x \cdot n = k$.

28. Suppose b, c are distinct points in $\mathbb{E}^d$. Let

$$H = \{x \in \mathbb{E}^d : \|x - b\| = \|x - c\|\}$$

(i.e. the set of all points equidistant from b, c).

Show that H is a hyperplane which passes through the midpoint of $[b, c]$ and which is orthogonal to the line $\overleftrightarrow{bc}$. Give an equation for H.

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Let $X = \{b_0, b_1, ..., b_r\}$ be an affinely independent subset of $\mathbb{E}^d$. The polytope conv(X) is called an **r-simplex**. (We allow $r < d$: think of a segment or triangle in space $\mathbb{E}^d$.)

29. Prove that a 1-simplex is a (proper) segment, and that a 2-simplex is a triangle.

30. Show that a d-simplex in $\mathbb{E}^d$ is not contained in any hyperplane H.

31. Let $\{b_0, b_1, ..., b_d\}$ be an affine basis for $\mathbb{E}^d$. Suppose $\|x - b_j\| = \|y - b_j\|$ for $0 \leq j \leq d$. Show that $x = y$. (See problem 28.)

C. **Isometries on** $\mathbb{E}^d$.

An **isometry** on $\mathbb{E}^d$ is any distance preserving map

$$S : \mathbb{E}^d \to \mathbb{E}^d$$

$$\|x - y\| = \|xS - yS\| \ , \ \ \forall x, y \in \mathbb{E}^d \ .$$

(Note that we shall normally write mappings on the right of their arguments: $xS := (x)S$, rather than $S(x)$.)

It is easy to check that the identity map $I : \mathbb{E}^d \to \mathbb{E}^d$ is an isometry.

32. (a) Show that any isometry is $1 - 1$.

    (b) We shall show below that isometries are onto, hence invertible. Try to prove this now. (See problem 1.)

33. (a) For each fixed vector $a \in \mathbb{E}^d$, the **translation**

    $$T_a : x \mapsto x + a$$

    is an isometry.

    (b) For which a does $T_a = I$?

    (c) The **central inversion**

    $$Z : x \to -x \ , \ \ \forall x \in \mathbb{E}^d$$

    is an isometry.

    (d) Any product (i.e. composition) of isometries is an isometry.

    (e) Identify the isometry $T_{-a}ZT_a$.

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We need to manufacture a richer supply of isometries.

34. Formulate a reasonable geometric definition for reflection in a hyperplane H. Verify that the following analytic definition meets your requirements.

    Let H be the hyperplane $x \cdot n = k$, with normal vector n. Then the **reflection** R in H is defined by

    $$R : x \mapsto x - \frac{2}{n \cdot n}(x \cdot n - k)n$$

    In particular, if $o \in H$, then $k = 0$ and

    $$R : x \to x - 2\frac{x \cdot n}{n \cdot n}n \ .$$

35.

(a) Verify that R is an isometry.

(b) Show that R fixes each point of H and interchanges the two half-spaces into which H decomposes $\mathbb{E}^d$.

(c) Show that $R^2 = I$. Thus R is invertible and $R = R^{-1}$.

(d) If $k = 0$, then R fixes the origin o and considered as a linear map has $\det(R) = -1$.

(e) If $x \notin H$, then H is the perpendicular bisector of the segment $[x, xR]$. (See problem 28.)

(f) For any $x_0, y_0 \in \mathbb{E}^d$, there exists a reflection mapping $x_0 \mapsto y_0$ (unique if $x_0 \neq y_0$).

36. Suppose $\|x_0 - x_1\| = \|y_0 - y_1\|$. Then there is a product of at most 2 reflections which maps both $x_0 \to y_0$ and $x_1 \to y_1$.

37. Suppose $X = \{x_0, ..., x_r\}$ and $Y = \{y_0, ..., y_r\}$ are affinely independent subsets of $\mathbb{E}^d$ with $\|x_i - x_j\| = \|y_i - y_j\|$ for all $0 \leq i, j \leq r$. Then there is an isometry S, in fact a product of at most $r + 1$ reflections, which maps each x_j to y_j, $0 \leq j \leq r$.

38. Suppose that an isometry S fixes each element of an affine basis $B = \{b_0, b_1, ...b_d\}$ for $\mathbb{E}^d$:

$$b_j S = b_j \ , \qquad 0 \leq j \leq d \ .$$

Then $S = I$. (Hint: see problem 31.)

39. (a) Every isometry S on $\mathbb{E}^d$ is a product of at most $d + 1$ reflections.

(b) If $B = \{b_0, ..., b_d\}$ and $C = \{c_0, ..., c_d\}$ are congruent affine bases for $\mathbb{E}^d$, i.e. if $\|b_i - b_j\| = \|c_i - c_j\|$ for $0 \leq i, j \leq d$, then there exists a unique isometry S mapping each $b_j \to c_j$.

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We now know that *all isometries are invertible mappings* on $\mathbb{E}^d$ and in fact constitute a group, which we denote $\mathcal{I}_d$.

The next result asserts that a 'local' isometry always extends to a 'global' isometry.

40. Let $X = \{x_j : j \in J\}$ and $Y = \{y_j : j \in J\}$ be two subsets of $\mathbb{E}^d$, such that $\|x_j - x_k\| = \|y_j - y_k\|$, for all $j, k \in J$. Then there is an isometry S on $\mathbb{E}^d$ such that $x_j S = y_j$, $\forall_j \in J$. Moreover S is unique if $\mathrm{aff}(X) = \mathbb{E}^d$.

(Hints: (a) Show that with no loss of generality we may assume $\mathrm{aff}(X) \subseteq \mathrm{aff}(Y)$. (b)Suppose $\{x_0, ..., x_m\}$ and $\{y_0, ..., y_m\}$ are affinely independent; then as in problem 37 we may assume $x_j = y_j$, for $0 \le j \le m$. Hence show that $x_{m+1} \notin \mathrm{aff}\{x_0, ..., x_m\}$ but with $y_{m+1} \in \mathrm{aff}\{y_0, ...y_m\}$ would contradict problem 31. (c)Show that $\mathrm{aff}(X) = \mathrm{aff}(Y)$ and use problem 31 again.)

41. (a) If R is reflection in the hyperplane $x \cdot n = k$, and S is any isometry, then $S^{-1}RS$ is also a reflection. Identify its (hyperplane) mirror.

    (b) Show that reflections R_1, R_2 commute if-f the corresponding normals are orthogonal: $n_1 \cdot n_2 = 0$.

42. If the isometry T maps o to b, then isometry S fixes o if-f $T^{-1}ST$ fixes b.

An isometry which fixes some point b is called an **orthogonal transformation**. Clearly, the collection of all isometries fixing b is a group, which by the previous problem is isomorphic, indeed conjugate to, the **orthogonal group**

$$\mathcal{O}_d = \{S \in \mathcal{I}_d \, : \, oS = o\} \ \ .$$

43. Each $S \in \mathcal{O}_d$ is a product of at most d reflections. (Hint: see problem 1; the first of the $d+1$ possible reflections is superfluous, since $oS = o$.)

Consider an isometry $S \in \mathcal{I}_d$, with $oS = b$, say. Applying the translation T_{-b} we find

$$ST_{-b} =: B \in \mathcal{O}_d \, .$$

Thus any isometry

$$S : \, x \longmapsto xB + b \ \ ,$$

can be viewed as the product of an orthogonal transformation (fixing the origin) and a translation.

44. Suppose

$$S : x \rightarrow xB + b$$

$$U : x \rightarrow xC + c \ \ ,$$

are two such isometries.

    (a) Show that $B \in \mathcal{O}_d$ and $b \in \mathbb{E}^d$ are uniquely determined by S.

    (b) Compute the unique orthogonal transformation and translation component for S^{-1} and for SU.

45. Show that

$$\mathcal{I}_d \simeq \mathcal{O}_d \ltimes \mathbb{R}^d$$

(a semi-direct product).

46. Show that every isometry S on $\mathbb{R}^d$ can be (uniquely) written as

$$S : x \to xB + b$$

where $b \in \mathbb{R}^d$ and B is an orthogonal $d \times d$ matrix (i.e. $BB^t = I$).

Notice that we may consider $\mathcal{O}_d$ to be a linear group (i.e. subgroup of $GL(\mathbb{E}^d)$).
We shall say $S : x \to xB + b$ is **direct** (sense-preserving) if $\det(B) = +1$, or **opposite** (sense-reversing) if $\det(B) = -1$.

47. (a) The isometry $S \in \mathcal{I}_d$ is direct (resp. opposite) if-f it is a product of an even (resp. odd) number of reflections. (See problem 1.)

(b) Show that the direct isometries constitute a subgroup ofindex 2 in $\mathcal{I}_d$.

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An isometry S with period 2 is called an **involution** (and is said to be involutary). Thus $S^2 = I$ but $S \neq I$. Examples include the central inversion Z (in o) and all reflections R.

48. Suppose S is an involutory isometry.

(a) For any $a \in \mathbb{E}^d$, S fixes the midpoint b of the segment $[a, aS]$.

(b) If b is the only fixed point of S then $S = T_{-b}ZT_b$, i.e. the central inversion
in b.

(c) The central inversion Z (at the origin o) is a product of reflections in d mutually perpendicular hyperplanes through o.

(d) Some conjugate of the involutary isometry S fixes o.

49. Let S be an involutory isometry.

(a) The fixed space $F = \{x \in \mathbb{E}^d : xS = x\}$ is an S-invariant affine subspace of $\mathbb{E}^d$.

(b) Let $b \in F$, which we suppose has dimension $d-r$. Then the subspace M through b, completely orthogonal to F, is an S-invariant r-dimensional subspace.

(c) S induces a central inversion on M.

(d) S is a product of reflections in r mutully perpendicular hyperplanes. (See problem 48 (c).)

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Let us investigate now *products of two reflections*, say

R_1 in the hyperplane $H_1 : x \cdot n_1 = k_1$,
R_2 in the hyperplane $H_2 : x \cdot n_2 = k_2$.

50. Recall that the above hyperplanes H_1 and H_2 are parallel if-f one is a translate of the other (problem 9).

(a) Show that H_1 is parallel to H_2 if-f the normals n_1, n_2 are linearly dependent.

(b) If H_1 is parallel to H_2, show that $R_1 R_2$ is a translation. The distance of the translation is twice the distance between the mirrors.

(c) Conversely, if T_b is any translation, show that $T_b = R_1 R_2$ is product of two reflections, in hyperplanes orthogonal to b. Given the latter requirement, show that either of R_1 or R_2 can be chosen arbitrarily.

51. Let R_1, R_2 be reflections in lines (i.e. 'hyperplanes') in $\mathbb{E}^2$, both passing through o.

If α is the angle from the first to second mirror, show that $R_1 R_2$ is a rotation through 2α, with centre o.

Because of this last problem, we say that the product of reflections R_1, R_2 in two intersecting hyperplanes H_1, H_2 is a **rotation**. The fixed space $H_1 \cap H_2$ is called the **axis** of rotation.

52. Suppose H_1 and H_2 are distinct, intersecting hyperplanes.

(a) Show that $F = H_1 \cap H_2$ is a $(d-2)-$dimensional affine subspace fixed by $S = R_1 R_2$.

(b) Let $L = \text{lin}\{n_1, n_2\}$ be the linear subspace spanned by the normals. Show that L is two dimensional.

(c) Let $b \in F$. Show that $M := L + b$ is a plane completely orthogonal to F at b. Show that H_1, H_2 meet M in lines inclined at an angle α satisfying

$$\cos\alpha = \frac{n_1 \cdot n_2}{\|n_1\| \; \|n_2\|} \quad .$$

(d) Show that each plane M is S-invariant and that S acts on M as a rotation through 2α.

(e) If S is a rotation with $(d-2)$-dimensional axis F and angle 2α, show that $S = R_1 R_2$, a product of reflections in hyperplanes H_1, H_2 inclined at angle α, and with $H_1 \cap H_2 = F$. Given this, show that either of H_1 or H_2 can be chosen arbitrarily.

The product of reflections in perpendicular mirrors is an involutory rotation, i.e. a **half-turn**. A **glide refection**, or just **glide**, is the product $G = RT$ of a reflection R with a translation T whose vector b is *parallel to the mirror* H for R. (Thus if R is reflection in the hyperplane H described by $x \cdot n = k$, then we in fact have $n \cdot b = 0.$).

53. Let $G = RT$ be the glide described just above.

(a) If $b = 0$, then the *degenerate* glide $G = R$ is actually an ordinary reflection.

(b) Show that $RT = TR$.

(c) Show that a **proper glide**, with $b \neq o$, has no fixed points.

(d) Show that $G = SR'$, where S is a half-turn whose axis is parallel to the mirror for the reflection R'. Given these requirements, R' can be replaced by reflection in any parallel mirror, adjusting the half-turn accordingly.

54. Any product RT of a reflection and translation is a glide. (Hint: resolve the vector $a = a_1 + a_2$ for T into components a_1 orthogonal to and a_2 parallel to the mirror for R. Apply problem 50(c) to T_{a_1} and T_{a_2}.)

D. **Isometries in the plane $\mathbb{E}^2$ and space $\mathbb{E}^3$ - first approach**

Here we shall classify all isometires in $\mathbb{E}^2$ or $\mathbb{E}^3$ in a routine, but self-contained, way. For a more insightful approach, particularly useful in classifying the regular polygons in ordinary space, see the next section (and chapter 1 of [1]). Refer to problems 43 and 47, concerning the orthogonal group $\mathcal{O}_d$, i.e. isometries in $\mathbb{E}^d$ which fix the origin o.

55. (a) Show that every orthogonal transformation the plane $\mathbb{E}^2$ is either a reflection or rotation.

    (b) Show that the orthogonal rotations in $\mathbb{E}^2$ form an abelian group $SO(2)$ isomorphic to $\mathbb{R}/Z$.

56. (a) (Euler) Every direct orthogonal transfunction in $\mathbb{E}^3$ is a rotation.

    (b) Show that $SO(3)$, the group of rotations fixing o in $\mathbb{E}^3$, is non-abelian.

    (c) Suppose that a rotation in $SO(3)$ is represented by an orthogonal matrix B. Show how to compute the rotation angle from trace(B).

Consider now a product $P = R_1R_2R_3$ of three reflections in $\mathbb{E}^2$ or $\mathbb{E}^3$. Suppose $(o)P^{-1} = b$ and let R_0 be reflection in the perpendicular bisector of $[o, b]$. (If $o = b$, take any mirror through o.) By problems 55(a) and 56(a), the direct isometry R_0P fixes o and so is a rotation, which can be factored as a product $R'R$ of two reflections in mirrors through o. In fact, we can freely choose R' and R to have perpendicular mirrors. Thus

$$R_0P = R'R$$

$$P = (R_0R')R = SR,$$

the product of a half-turn S and reflection R.

57. In $\mathbb{E}^2$, the product of three reflections is always a glide, perhaps degenerating to a reflection (see problem 53(c).)

58. **Theorem**.

    (a) In the plane $\mathbb{E}^2$, every isometry is either a reflection, rotation, translation or glide.

    (b) Every isometry in $\mathbb{E}^2$ is the product of two involutory isometries.

Consider again $P = SR$ above. In $\mathbb{E}^3$, the axis for the half-turn S is a line L (which may or may not intersect the mirror H for R. In any case, $S = R_4R_5$, a product of reflections in orthogonal mirrors H_4, H_5, where we may freely choose $H_5 \perp H$. Thus

$$P = R_5U = UR_5$$

where $U = R_4R$ is either a translation, when L is parallel to H, or a rotation with axis $M \perp H_5$, when L meets H.

In the latter case, P is called a **rotatory reflection**, i.e. the product of a rotation and reflection, whose mirror is perpendicular to the axis of rotation.

59. Explain why the central inversion is a rotatory reflection.

60. Summarize the above discussion: in $\mathbb{E}^3$ every product of three reflections is either what or what?

A product $P = R_1R_2R_3R_4$ of four reflections can be similarly analyzed. Let $(o)P = b$, so $P' = PT_{-b}$ is a direct isometry fixing o, namely a rotation S with axis L. Suppose $b = b_1 + b_2$, where $b_2 \in L$ and $b_1 \cdot b_2 = 0$. Thus $T_{b_1} = R_6R_7$, where R_6, R_7 are suitable reflections, whose mirrors are orthogonal to b_1, and where the mirror for R_6 contains L. But then $S = R_5R_6$ for a suitable reflection R_5. In short,

$$\begin{aligned} P &= ST_b \\ &= ST_{b_1}T_{b_2} \\ &= R_5R_6R_6R_7T_{b_2} \\ &= (R_5R_7)T_{b_2} \\ &= S'T_{b_2} \ , \end{aligned}$$

the product of a rotation S' whose axis L' is parallel to the vector b_2 for the translation T_{b_2}. Such a direct isometry is called a **screw** (or **screw displacement**).

61. **Theorem.**

(a) Every isometry in $\mathbb{E}^3$ is a reflection, rotation, translation, glide, rotatory-reflection or screw.

(b) Every isometry in $\mathbb{E}^3$ is a product of two involutory isometries.

E. **Isometries in the plane $\mathbb{E}^2$ and space $\mathbb{E}^3$ - a second approach using regular polygons**

A by-product of the classification of isometries in $\mathbb{E}^2$ and $\mathbb{E}^3$ (see problems 58 and 61 in the previous section) is the observation that each such isometry is the product of two involutary isometries. In fact, this is true in general and can be proved with a careful analysis of the eigenspaces of a general orthogonal matrix, without first having a full classification of the various isometries (see [1, page 3]).

Theorem. In $\mathbb{E}^d$, with $d \geq 2$, each isometry can be expressed as a product of two involutary isometries.

62. Find a proof for this theorem.

In [1, ch. 1] Coxeter uses this theorem as a starting point for a clssification of isometries which is keyed to the notion of a regular polygon.

We define a **polygon** $\mathcal{P}$ in $\mathbb{E}^d$ to be a sequence of points or **vertices**

$$\ldots, a_{-1}, a_0, a_1, a_2, \ldots$$

joined in successive pairs by the segments or **edges**

$$\ldots, [a_{-1}, a_0], [a_0, a_1], [a_1, a_2], \ldots$$

(say $e_j := [a_j, a_{j+1}]$).

This list of vertices could be finite, say $a_0, a_1, \ldots, a_{p-1}$, in which case we take subscripts to be residues $\pmod p$. Thus when there are p distinct vertices, we obtain a p-gon with edges $e_0 = [a_0, a_1]$, $e_1 = [a_1, a_2]$, ending with the return edge $e_{p-1} = [a_{p-1}, a_0]$. On the other hand, if the a_j's are distinct for all $j \in \mathbb{Z}$, we obtain the (infinite) **apeirogon** $\{\infty\}$.

Naturally, we agree that the **dimension** of $\mathcal{P}$ is determined by the dimension of the affine hull of the vertex set. Thus a polygon is **linear** if it sits inside a line, **planar** if in a plane, and **skew** otherwise.

We say that $\mathcal{P}$ is **regular** if for some isometry S we have

$$a_j = a_0 S^j \ , \ \ \forall j \in \mathbb{Z} \ .$$

Hence $\mathcal{P}$ is regular if there is an isometry which induces the cyclic permutation

$$(\ldots, a_{-1}, a_0, a_1, a_2, \ldots)$$

(on the vertex set).

63. Typically one also demands that the base vertex a_0 not be fixed by S. Why? What is p in this case?

64. Describe the possible regular p-gons when S is an involution.

65. Suppose $\mathcal{P}$ is the regular polygon generated by the isometry S. Prove that there are *involutary* isometries R_0, R_1 which induce the following permutations on the vertex set:

$$R_0 : \ldots (a_0, a_{-1})(a_1, a_{-2})(a_2, a_{-3}) \ldots$$

$$R_1 : \ldots (a_1, a_{-1})(a_2, a_{-2})(a_3, a_{-3}) \ldots$$

(Hint: use problem 40).

More precisely, show that $S = R_0 R_1$, at least on the affine hull of the vertex set, and that R_1 fixes the base vertex a_0.

66. Why does the previous problem not quite solve problem 62 ? Perhaps one can enhance this approach to produce a complete proof.

67. [1, page 4] Show that $\mathcal{P}$ is regular if and only if for each positive integer k we have

$$\|a_0 - a_k\| = \|a_1 - a_{k+1}\| = \|a_2 - a_{k+2}\| = \ldots .$$

68. [1, page 4] A pentagon is regular if and only if it is equilateral and equiangular.

69. [1, page 4] A polygon is regular if and only if its edges are all equal and its angles of each kind are all equal (Schoute, 1902). For example, a skew polygon in $\mathbb{E}^3$ has two kinds of angles: the usual angle $a_{j-1}a_j a_{j+1}$ at a vertex, and a dihedral angle $a_{j-1}(a_j a_{j+1})a_{j+2}$ at an edge e_j.

70. [1, page 4] Describe a regular skew pentagon in $\mathbb{E}^4$.

****

Since the isometry S used above is quite general, we observe that the classifications of isometries and regular polygons, say in $\mathbb{E}^2$ and $\mathbb{E}^3$, go hand in hand and can be achieved by examining all posssible pairs of involutions R_0, R_1. Moreover, we deduce from problem 49 that any involutary isometry is determined by its fixed space F, and so is classified merely by the dimension of this fixed space.

The actual possibilities for a regular polygon $\mathcal{P}$ in $\mathbb{E}^2$ or $\mathbb{E}^3$ now fall out quite neatly. We summarize the discussion in [1, §1.6 – 1.8] by a sequence of problems:

71. Suppose R_1 is central inversion.

(a) Show that edges $e_0 = [a_0, a_1]$ and $e_{-1} = [a_{-1}, a_0]$ are collinear.

(b) Show that $\mathcal{P}$ is the one dimensional apeirogon $\{\infty\}$, consisting of infinitely many equally spaced points on a line, joined consecutively. Thus $\mathcal{P}$ is *linear*.

(c) Show that R_0, R_1 act on this line by reflection (= central inversion!), and S by translation.

72. Suppose R_0 is either reflection or central inversion.

(a) Show that the planes $a_0 a_{-1} a_{-2}$ and $a_{-1} a_0 a_1$ coincide, so that $\mathcal{P}$ must be *planar*.

Hence we may

- assume that R_0 is a reflection or half-turn in this plane
- assume that R_1 is a reflection in this plane (else R_1 is a half-turn = central inversion, and we are returned to the case in problem 71)
- assume further that if R_0 is a reflection, then the mirrors for R_0, R_1 intersect, say at o (the case of parallel mirrors returns us again to problem 71)

(b) So assume that R_0, R_1 are reflections whose mirrors intersect at o, and let $\alpha := \angle a_{-1}\, o\, a_0$. Describe $\mathcal{P}$ when α is incommensurable with π. When $\alpha = 2d\pi/p$, for coprime integers d and p, we obtain a p-gon of density d inscribed in a circle. We denote this finite regular polygon by $\{p/d\}$. If $p < 2d$, we usually write $\{p/(p-d)\}$ instead: thus $\{3/2\}$ really describes an equilateral triangle.

When is $\{p/d\}$ convex?

(c) The final planar case has R_0 a half-turn and and R_1 a reflection, so that S is a glide: see problem 53(d). Show that $\mathcal{P}$ is an infinite regular **zig-zag**, another realization of the combinatorial polygon $\{\infty\}$. When does this realization 'degenerate' into the linear apeirogon?

The above analysis covers all possibilities in the plane $\mathbb{E}^2$. The remaining possibilities in space $\mathbb{E}^3$ have R_0 a half-turn and R_1 either a reflection or half-turn.

73. Suppose that R_0 is a half-turn with axis L and R_1 a reflection with mirror H in $\mathbb{E}^3$.

(a) Suppose L is parallel to H. Show that $S = R_0R_1$ is a glide (problem 53); furthermore, any choice of $a_0 \in H$ again provides a zig-zag lying in the plane through a_0 orthogonal to H (so we return to problem 72(c)).

(b) Suppose L meets H, say at o. Then $S = R_0R_1$ is akin to a 'spherical glide'. In fact, S is a rotatory-reflection (see the discussion following problem 58). The polygon $\mathcal{P}$ is now a 'spherical zig-zag', whose vertices lie alternately on two parallel 'small' circles. To see this another way, show that $S = UR$, the product of a rotation U through angle α with reflection in a mirror orthogonal to the axis of rotation.

If $\alpha = d\pi/p$, for coprime integers d, p, then S has period $2p$. The polygon $\mathcal{P}$ is a skew $2p$-gon, whose alternate vertices are inscribed in the two small circles. In fact, each set of alternate vertices gives a p-gon inscribed in its small circle. Further description requires some care; it is natural to distinguish two subcases:

- If p is odd and d is even (still coprime), show that $\mathcal{P}$ is a **prismatic** polygon, whose vertices are those of a right prism with base $\{p/(\frac{1}{2}d)\}$, and whose $2p$ edges are all lateral face diagonals of this prism.

- In all other cases, $\mathcal{P}$ is an **anti-prismatic** polygon, whose vertices are those of an antiprism with base $\{p/d\}$, and whose $2p$ edges are simply all edges of this antiprism.

74. Suppose that R_0 is a half-turn with axis L_0 and R_1 another half-turn with axis L_1 in $\mathbb{E}^3$.

    (a) If L_0 and L_1 intersect or are parallel, the plane containing them is S-invariant, so that $\mathcal{P}$ is again planar (or even linear).

    (b) So suppose L_0 and L_1 are skew lines with unique common perpendicular line M. Show that S is a screw with axis M (see the discussion preceding problem 61).

    If $a_0 \notin M$, and L_0 and L_1 are not orthogonal, then $\mathcal{P}$ is skew and is inscribed in a helix. Hence $\mathcal{P}$ is called a **helical polygon**. Such polygons come naturally in **chiral** pairs (i.e. left- and right-handed versions).

75. [1, page 6] Every finite regular skew polygon in $\mathbb{E}^3$ has an even number of vertices.

76. [1, page 6] A skew pentagon in $\mathbb{E}^3$ cannot be both equilateral and equiangular.

77. [1, page 6] Can a skew hexagon in $\mathbb{E}^3$ be both equilateral and equiangular, but not regular?

F. Affine Irreducibility.

The upshot of the discussion around problem 44 is that every isometry S is an affine isomorphism. Thus S maps any r-dimensional affine subspace to another of the same dimension. In particular, if S maps L to some translate

$$(L)S = L + c \ ,$$

then S permutes the members of

$$\hat{L} := \{L + x \, : \, x \in \mathbb{E}^d\} \ ,$$

the family of all translates of L. Considering the usual way of projectifying $\mathbb{E}^d$, we might say that S fixes an $(r-1)$-space $\hat{L}$ at infinity. In particular, this happens if L is actually S-invariant.

78. If the isometry S maps the r-space L to a translate, then S maps any orthogonal $(d-r)$-space M to a translate. Thus $\hat{L}$ being invariant implies that $\hat{M}$ is invariant.

79. What is the space at infinity when $L = \{a\}$ is a point?

80. How does any translation T act on the various spaces at infinity?

81. Suppose
$$S : x \rightarrow xB + b \ ,$$
where $B \in \mathcal{O}_d$. (Thus B is an orthogonal transformation fixing o.) Then $\hat{L}$ is S-invariant if–f its linear direction space $L - L$ is B-invariant.

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In fact, suppose now that the linear subspaces L_1, L_2 are fixed orthogonal complements at o, and let

$$\pi_1 : \mathbb{E}^d \rightarrow L_1$$

$$\pi_2 : \mathbb{E}^d \rightarrow L_2$$

be the corresponding (linear) orthogonal projections. (If we really must represent the situation when affine subspaces are orthogonal at a, then conjugate appropriately by the translation T_a.)

82. Observe that

$$z = z\pi_1 + z\pi_2 \ , \quad \forall z \in \mathbb{E}^d \ .$$

83. Suppose $S \in \mathcal{I}_d$ is any isometry which maps L_1 (likewise L_2) to some translate. Define a mapping S_j on L_j by

$$\begin{aligned} S_j : \ L_j &\to L_j \\ x &\to xS\pi_j \ . \end{aligned}$$

(a) Show that S_j is a well-defined isometry on L_j.

(b) Show that

$$zS = z\pi_1 S_1 + z\pi_2 S_2 \ , \quad \forall z \in \mathbb{E}^d \ .$$

(Thus S may be reconstituted in a natural way from the induced mappings on L_1 and L_2.)

As a temporary notation, denote the isometry group on L_j by $\mathcal{I}(L_j)$.

84. Suppose Γ is a subgroup of $\mathcal{I}_d$ which fixes $\hat{L}_1$ (hence also $\hat{L}_2$) for the above orthogonal linear subspaces L_1, L_2 (i.e. each $S \in \Gamma$ maps L_1 to some translate). Show that the maps

$$\begin{aligned} \varphi_j : \ \Gamma &\to \mathcal{I}(L_j) \\ S &\to S_j \end{aligned}$$

are group homomorphisms.