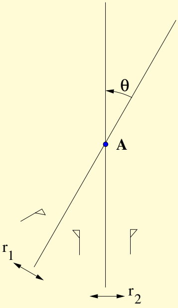
Reflection and Rotation groups in Ordinary Space

Barry Monson (UNB)
Summer School on Symmetries of Combinatorial Structures
Cuernavaca, July, 2012

(supported in part by the NSERC of Canada)

Isometries in the Euclidean plane $\mathbb{E}^2$

In the plane $\mathbb{E}^2$, the product of reflections r_1, r_2 (in intersecting lines) is



$r_1 r_2 =$ rotation through angle 2θ about centre A

Note:

- compose mappings *left-to-right*
- the rotation $r_1 r_2$ has finite period q if $\theta = \pi/q$, so usually period $= \infty$.
- reflections reverse orientation; determinant $= -1$
- rotations preserve orientation; determinant $= +1$
- in space $\mathbb{E}^3$ $r_1 r_2$ is still a rotation (but now with a linear axis) Why?
- think of the action in a plane perpendicular to the two planar mirrors for r_1 and r_2

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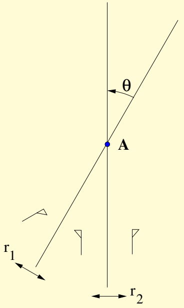

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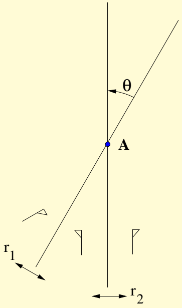

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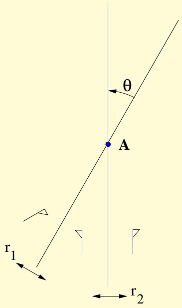

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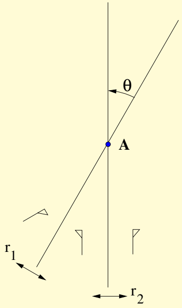

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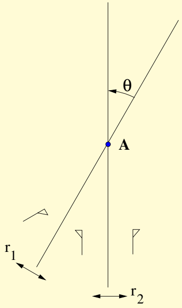

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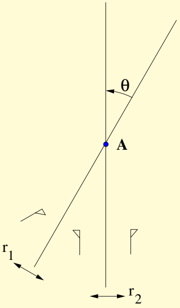

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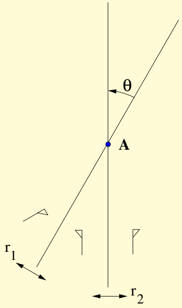

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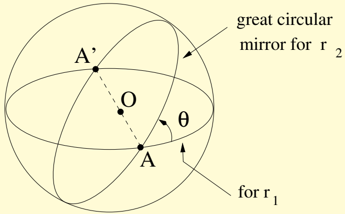
mirrors in $\mathbb{E}^3$ will intersect in some point O so we can follow the action in a unit sphere $\mathbb{S}^2$ centred at O.



- mirrors become *great circles*
- for sure $r_1 r_2$ is a rotation through 2θ at centre A.
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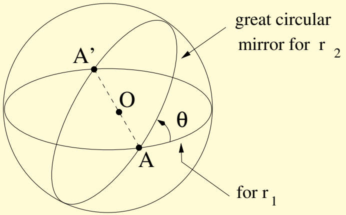

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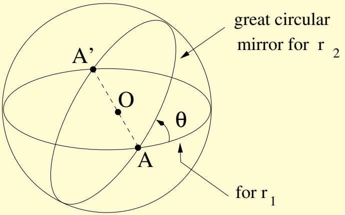

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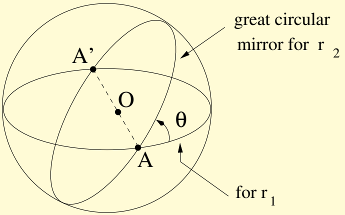

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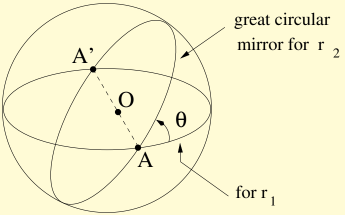

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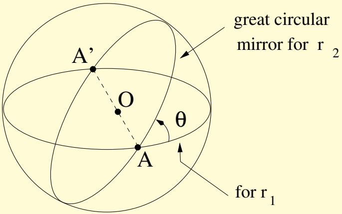

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Reflection groups G

Let G be any finite *group generated by reflections* in $\mathbb{E}^3$.

Familiar examples: symmetry group G of a Platonic solid

Algebraic Properties of G:

- the map
$$\begin{aligned} G &\rightarrow \{\pm 1\} \\ g &\mapsto \det(g) \end{aligned}$$
is a homomorphism.
- The kernel G^+ contains all products of an even number of reflections. G^+ has index 2 in G.
- In $\mathbb{E}^3$ these kernel symmetries are all rotations, including identity 1.
- the other coset contains all reflections and usually other symmetries, namely *rotatory reflections* = 'spherical glides'.

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- any point $P \in \mathbb{E}^3$ has a finite G-orbit
- but $O =$ centroid of that orbit is G-fixed.
- so can use unit sphere $\mathbb{S}^2$ centred at O
- the mirrors of <u>all</u> reflections in G pass through O and subdivide $\mathbb{E}^3$ into various 'conical regions' all sharing the apex O and each bounded by certain mirrors ▸ See a cube
- we can manufacture a *kaleidoscope* by lining one such region with real physical mirrors
- the cones subdivide $\mathbb{S}^2$ into spherical polygons.

The cube is typical ...

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The cube $\{4, 3\}$ and its group $G = [4, 3]$

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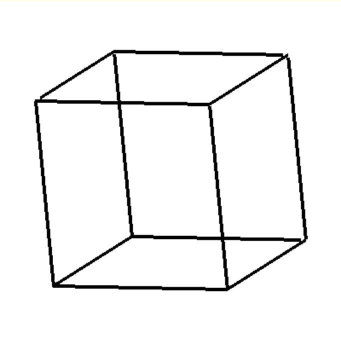

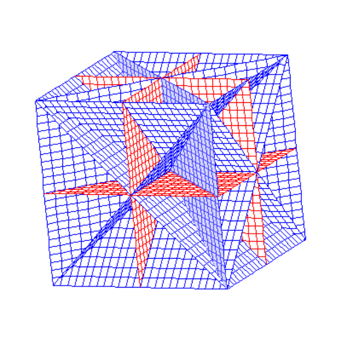

Schläfli symbol = $\{4, 3\}$ means all *facets* are squares $\{4\}$, and all *vertex-figures* are equilateral triangles $\{3\}$ (simply put, vertices have degree 3)

The cube has 9 mirrors of symmetry
3 red (parallel to facets)
6 blue (through opp. edge pairs)

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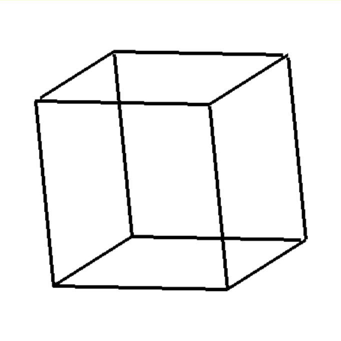

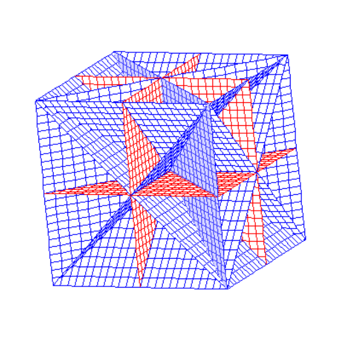

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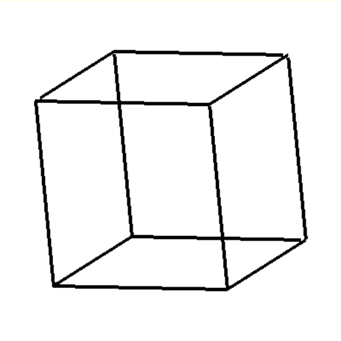

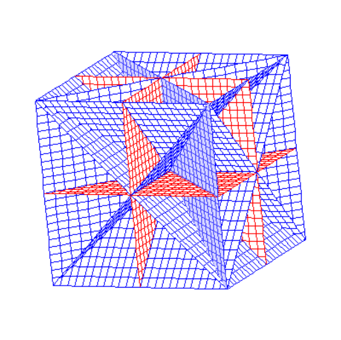

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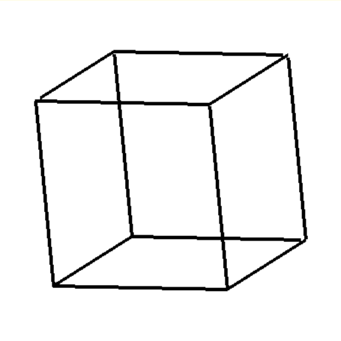

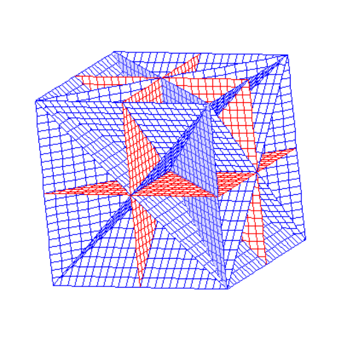

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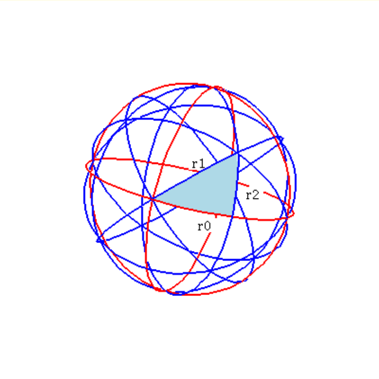

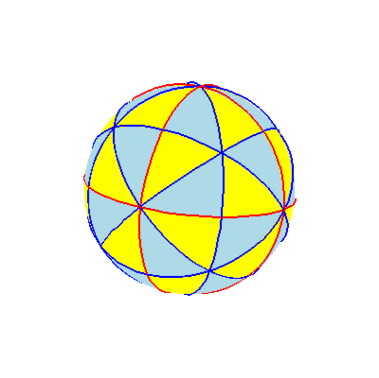

Pick one 'base' cone. It meets the sphere in a *base spherical triangle* with angles $\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$. It does look like a '*flag*'.

The whole sphere is tiled by 48 such triangles. The order of group G must equal 48.

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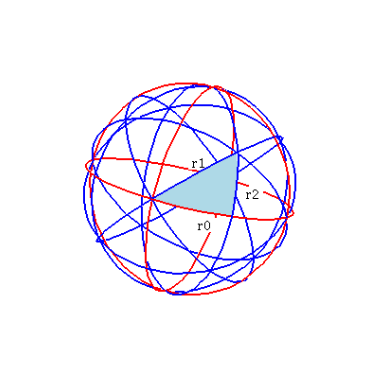

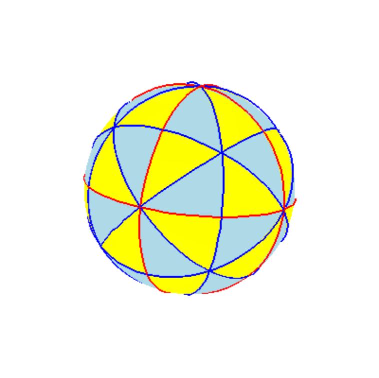

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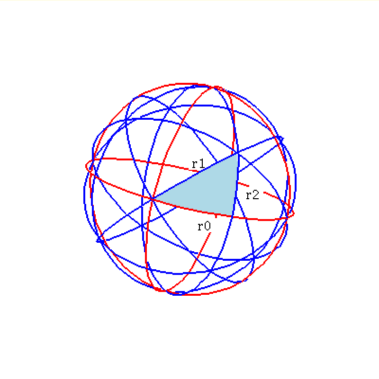

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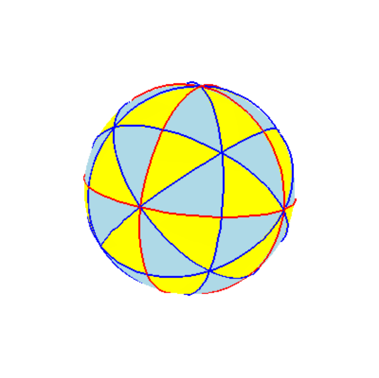

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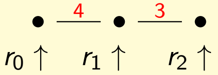

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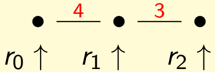

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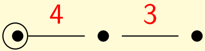
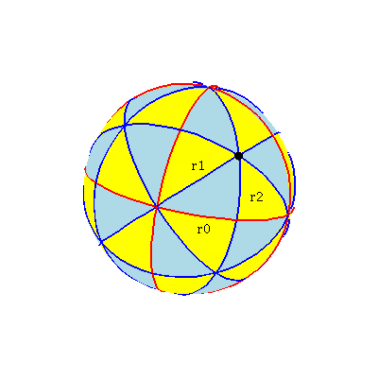
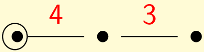


to summarize *Wythoff's construction* for a regular or Archimedean poyhedron whose symmetry group is G ($= [4, 3]$ in this example).

Wyrhoff's construction





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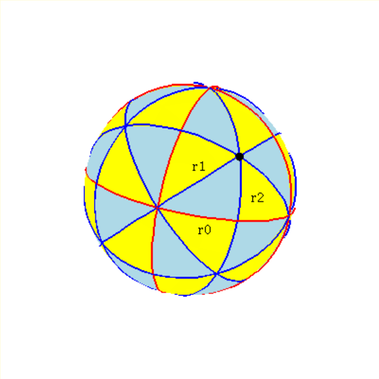

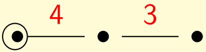

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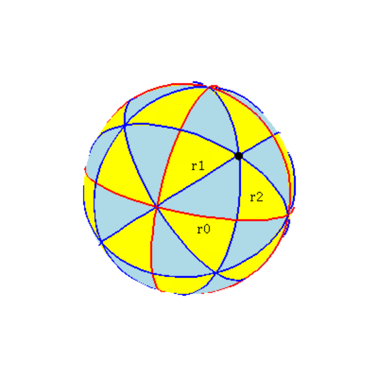

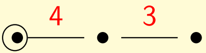

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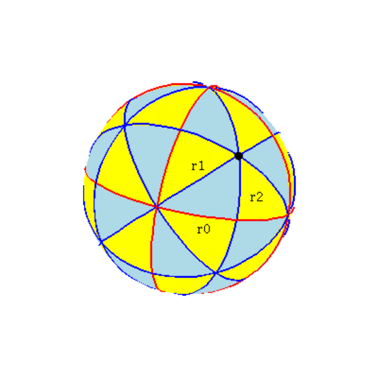

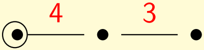

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UNB

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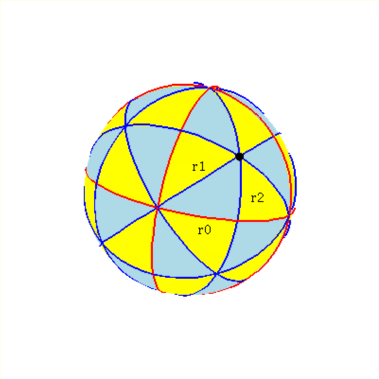

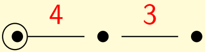

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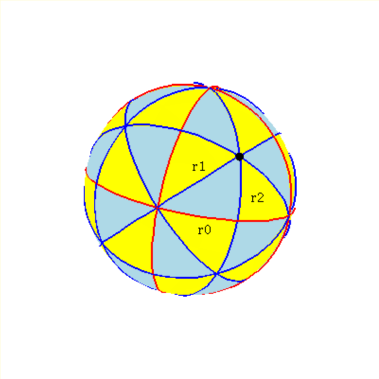

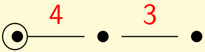

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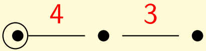

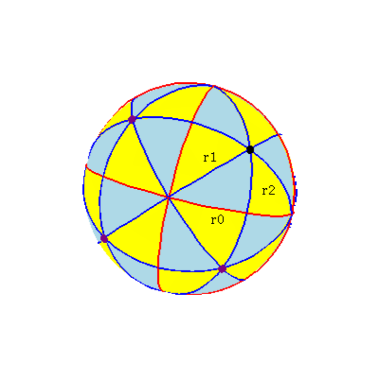

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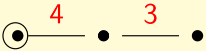

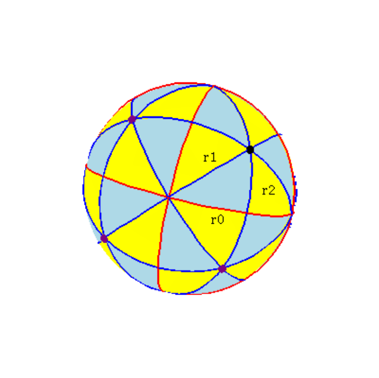

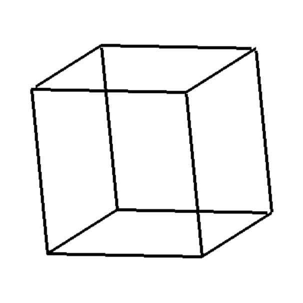

The *cube* of course!

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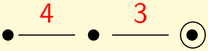

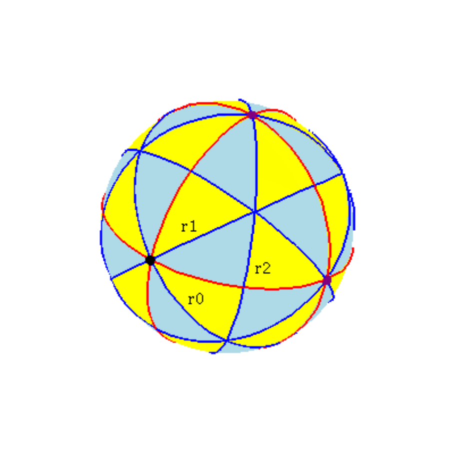

UNB

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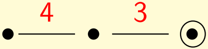
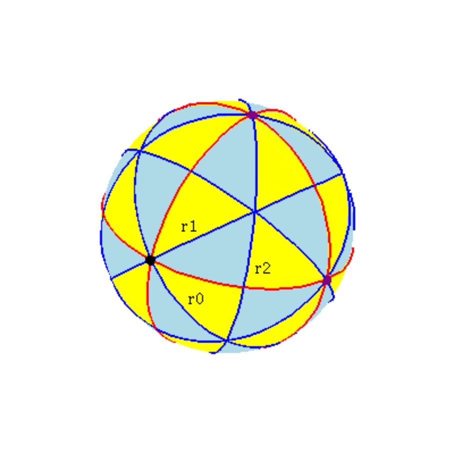
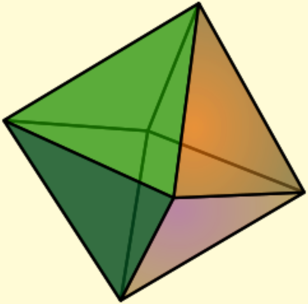


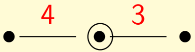


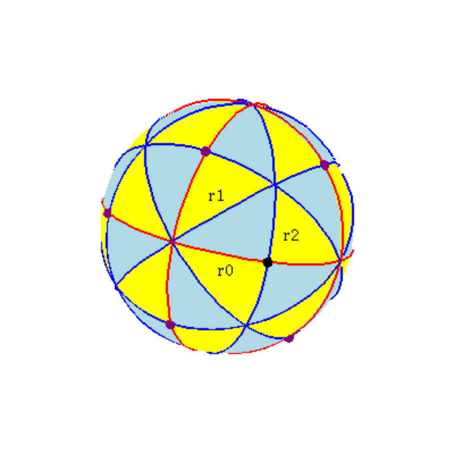


(courtesy Wikipedia)

The *octahedron* is *dual* to the cube.

Identify





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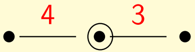
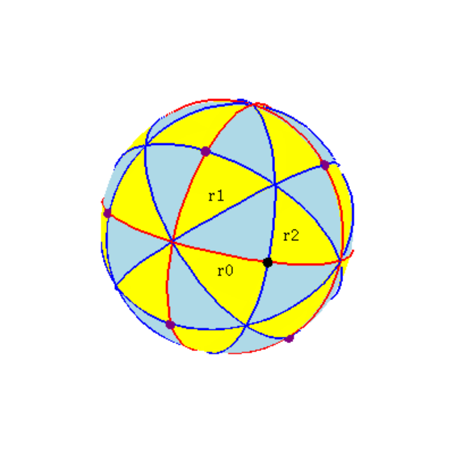
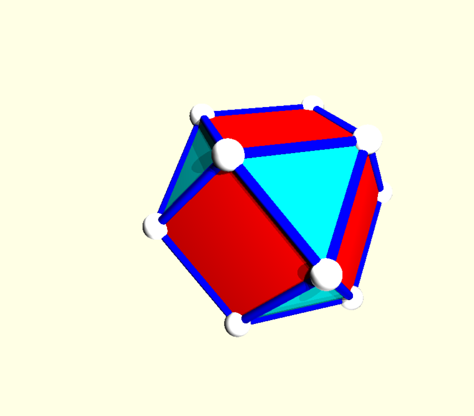
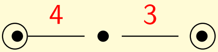


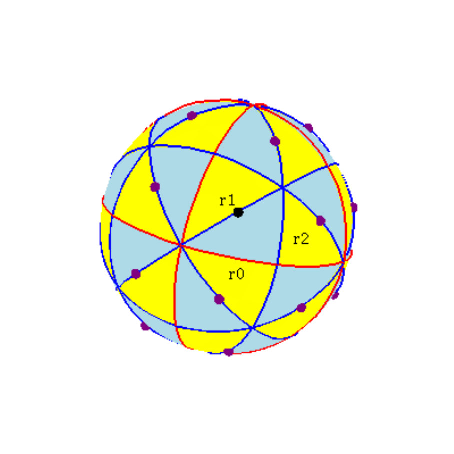




cuboctahedron

Identify





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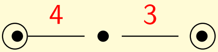
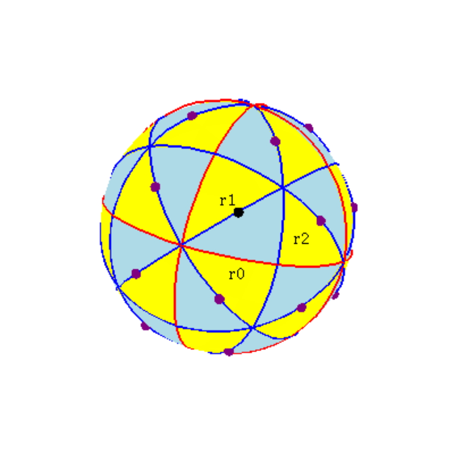
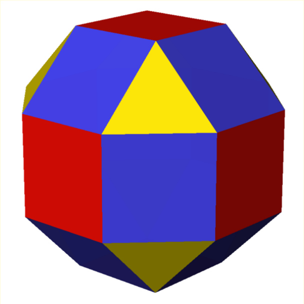


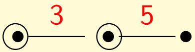




rhombicuboctahedron
(again courtesy Wikipedia)

Identify 

In Ryan Oulton's Maple program the diagram



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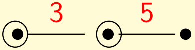

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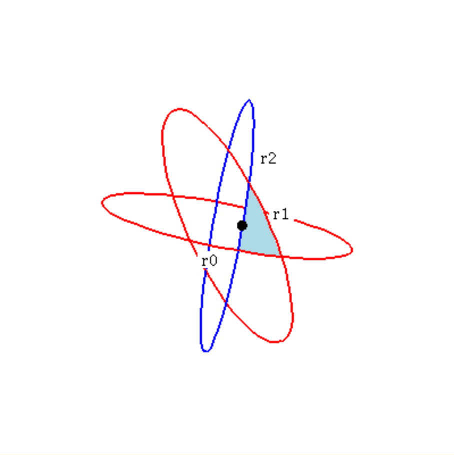
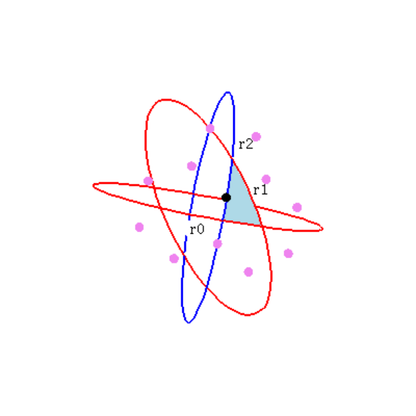


Here you see the initial vertex...

And here is

some of the orbit:



What do we get?

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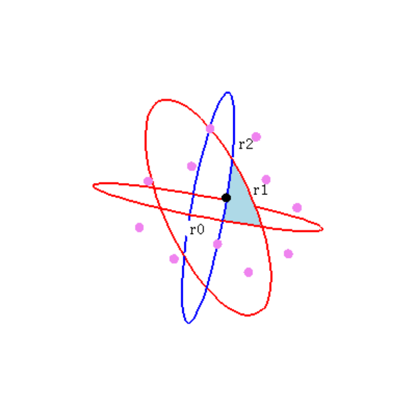
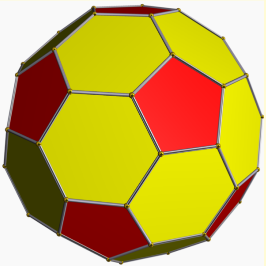


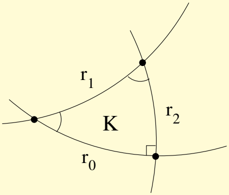


The *truncated icosahedron* (from Wikipedia), i.e a soccer ball.

What do we get?

General reflection groups G in $\mathbb{E}^3$

Let K be one of the "minimal" regions cut out on $\mathbb{S}^2$ by the set of mirrors in G.



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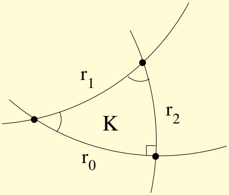

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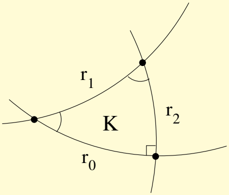

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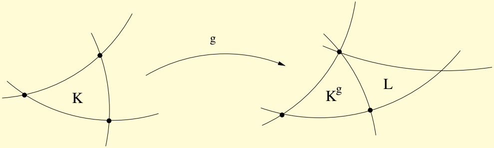
How is G generated?

For each $g \in G$, the image K^g of K under the symmetry g must be another region; suppose region L is adjacent to K^g:



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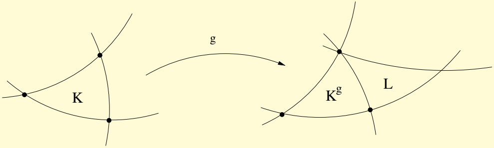

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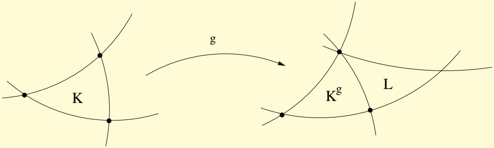

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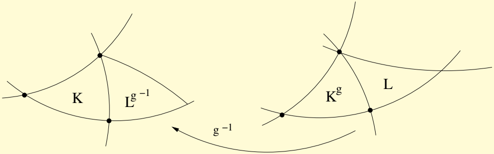

Conclude: any region L adjacent to K^g must equal $K^{r_j g}$ for some j.

UNB

There are many consequences!

- any $g \in G$ can be written $g = r_{j_m} \cdots r_{j_1}$, i.e. G is generated by reflections $r_0, \ldots, r_{n-1}$ in mirrors bounding the chosen fundamental region K.
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- show each interior angle of K has the form π/p for some integer $p \geq 2$. Call these angles $\pi/p_0, \ldots, \pi/p_{n-1}$.
- use angular excess to show K has area $\pi[\frac{1}{p_0} + \cdots + \frac{1}{p_{n-1}} - (n-2)]$.
- since this is positive and since each $\pi/p_j \leq \pi/2$ we get

$$0 < \frac{n}{2} - (n-2)$$

which shows

- K has at most 3 sides, so $n = 2$ or 3, or even 1 as a 'degenerate' case. Further, when $n = 2$ and K is a 'lune', we must have $\pi/p_0 = \pi/p_1$. Put all this together to get

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- K has at most 3 sides, so $n = 2$ or 3, or even 1 as a 'degenerate' case. Further, when $n = 2$ and K is a 'lune', we must have $\pi/p_0 = \pi/p_1$. Put all this together to get

Next

- show each interior angle of K has the form π/p for some integer $p \geq 2$. Call these angles $\pi/p_0, \ldots, \pi/p_{n-1}$.
- use angular excess to show K has area $\pi[\frac{1}{p_0} + \cdots + \frac{1}{p_{n-1}} - (n-2)]$.
- since this is positive and since each $\pi/p_j \leq \pi/2$ we get

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The Classification of finite reflection groups G in $\mathbb{E}^3$

G belongs to one of the following classes:

1. $G = \langle r_0 \rangle$ is generated by one reflection and has order 2. In this case K is a hemisphere.

2. $G = \langle r_0, r_1 \rangle$ is a dihedral group $\mathbb{D}_p$ for some $p \geq 2$. Here G has order $2p$ and K is a lune bounded by semicircles with polar angle π/p.

3. $G = \langle r_1, r_2, r_3 \rangle$ is generated by three reflections whose mirrors bound a spherical triangle K. The actual subcases are

• $(p_1, p_2, p_3) = (2, 2, p)$ for any integer $p \geq 2$. Here G has order $4p$ and can serve as the symmetry group of of a uniform p-gonal right prism.

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and more: the groups of the regular polyhedra

• $(p_1, p_2, p_3) = (2, 3, 3)$. Here G has order 24, is isomorphic to the symmetric group S_4 and serves as the symmetry group of the regular tetrahedron $\{3, 3\}$.

• $(p_1, p_2, p_3) = (2, 3, 4)$. Here G has order 48 and can serve as the symmetry group of the cube $\{4, 3\}$ or regular octahedron $\{3, 4\}$. (Here $G \simeq S_4 \times C_2$.)

• $(p_1, p_2, p_3) = (2, 3, 5)$. Here G has order 120 and can serve as the symmetry group of the regular dodecahedron $\{5, 3\}$ or regular icosahedron $\{3, 5\}$. (G is not isomorphic to S_5; instead $G \simeq A_5 \times C_2$.)

We note that the order of the symmetry group $[p, q]$ for the regular polyhedron $\{p, q\}$ is

$$\frac{4}{\frac{1}{p} + \frac{1}{q} - \frac{1}{2}} = \frac{8pq}{4 - (p-2)(q-2)}.$$

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