Monodromy, not Dromedary

Barry Monson (UNB)
Workshop on Abstract Polytopes
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Exercise (not Problem) 1

What do *monodromy* and *dromedary* have in common?

Tentative solution:
The term 'monodromy' may first have been used by Cauchy in *Exercices d'analyse et de physique mathématique*, vol. IV (1847), page 325, from $\mu o \nu o$ (mono), meaning 'single', of course, with $\delta \rho \acute{o} \mu o \varsigma$, relating to 'racecourse' or 'running'.

Cauchy was concerned with a complex function having *one* value when we attempt to continue it analytically by *running* along a curve.

As far as I know, Cauchy had little interest in dromedaries (14th C), which are fast *running* camels.

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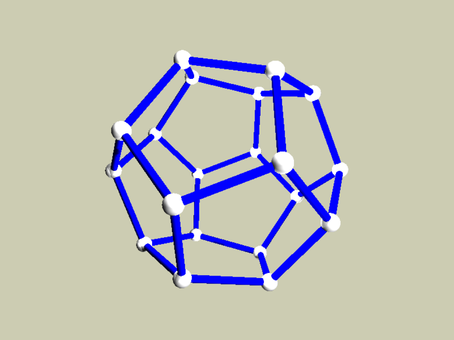
Example 1. The regular dodecahedron $\mathcal{D}$



Here $\Gamma(\mathcal{D})$ is the Coxeter group

$$H_3 = \bullet \overset{5}{\text{———}} \bullet \overset{3}{\text{———}} \bullet$$

of order 120.

The flags correspond exactly to the triangles in a barycentric subdivision of the surface of $\mathcal{D}$.

Here is part of that $\Rightarrow$

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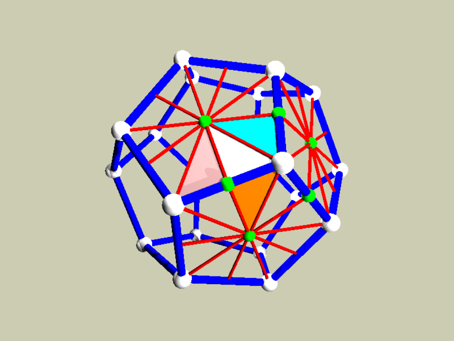
A base flag for $\mathcal{D}$, adjacent flags and generators



By transitivity, pick any base flag $= \Phi$ [white]
Then
0-adjacent flag $=: \Phi^0$ [pink]
1-adjacent flag $=: \Phi^1$ [cyan]
2-adjacent flag $=: \Phi^2$ [orange]
For $i = 0, 1, 2$, there is a unique automorphism

$$\rho_i : \Phi \mapsto \Phi^i \ .$$

Then $\Gamma(\mathcal{D}) = \langle \rho_0, \rho_1, \rho_2 \rangle$.
Can think reflections $\Rightarrow$

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Regular polytopes and string C-groups

Schulte (1982) showed that the regular d-polytopes $\mathcal{Q}$ correspond exactly to the *string C-groups*

$$\langle \rho_0, \ldots, \rho_{d-1} \rangle \ \ [\simeq \Gamma(\mathcal{Q})],$$

which we often study in their place. ▸ more

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Now DESTROY the polytope!

Consider any d-polytope $\mathcal{Q}$, not necessarily regular. For each flag Φ of $\mathcal{Q}$ and $i = 0, \ldots, d-1$, there is a unique *i-adjacent* flag Φ^i.

The mapping $s_i : \Phi \mapsto \Phi^i$ defines an involutory bijection s_i on the set $\mathcal{F}(\mathcal{Q})$ of all flags.

Defn. The *monodromy group* of $\mathcal{Q}$ is $\mathrm{Mon}(\mathcal{Q}) := \langle s_0, \ldots, s_{d-1} \rangle$.

(For maps, Steve Wilson [1994] calls this the "connection group".)

It is easy to check that $s_i^2 = 1$ and that $(s_i s_j)^2 = 1$, for $|j - i| > 1$, so $\mathrm{Mon}(\mathcal{Q})$ is an *sggi = string group generated by involutions*,

but

can it fail the intersection condition needed to to be a string C-group = aut. group of regular d-poly?

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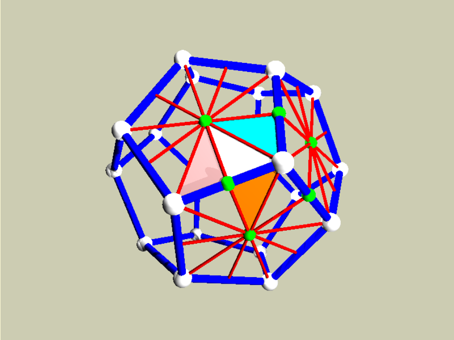
Example 1 - more on the regular dodecahedron $\mathcal{D}$



Note how seemingly destructive such flag swaps are.
(Think Rubik.)
Even so, here we do have

$$\mathrm{Mon}(\mathcal{D}) \simeq \Gamma(\mathcal{D})\ .$$

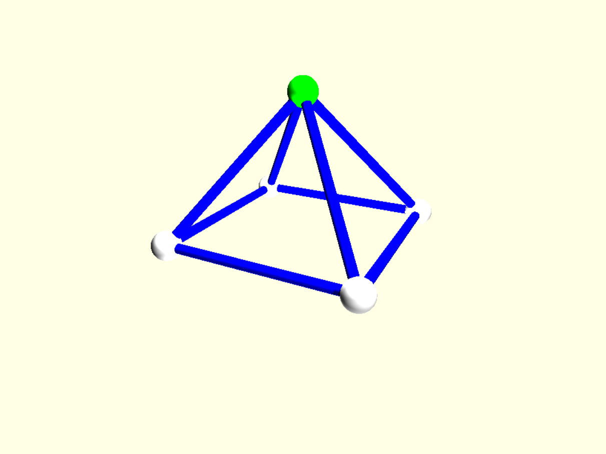
Theorem[ours in high rank]
For any abstract regular d-polytope $\mathcal{P}$,

$$\mathrm{Mon}(\mathcal{P}) \simeq \Gamma(\mathcal{P})\ .$$

See *Mixing and Monodromy of Abstract Polytopes*, Monson, Pellicer and Williams, coming soon.

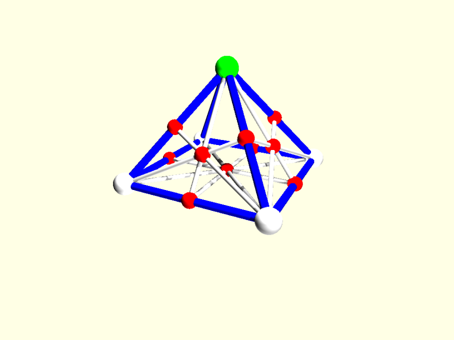
Example 2. The 4-gonal pyramid $\mathcal{E}$ is *not* regular

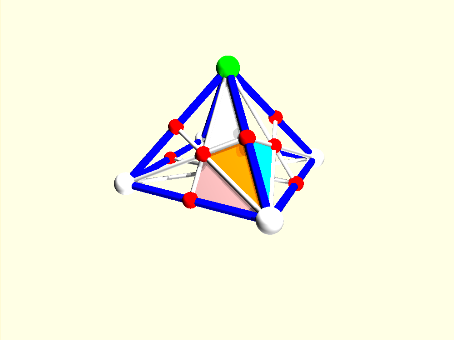
You can see that $\Gamma(\mathcal{E})$ has order 8. Guess the order of its monodromy group . . .



Example 2, continued

Here is a bit of the barycentric subdivison (left) with a few flags (right). Start flipping!





Example 2, continued

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In fact, the monodromy group of this pyramid has order

$$2^{11} \cdot 3 = 6144 \; .$$

It follows from theorems coming up that the unique minimal regular cover of the pyramid is a finite, self-dual regular map of Schläfli type $\{12, 12\}$ and genus 257.

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