Problems for the Workshop on Abstract Polytopes,
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Barry Monson, University of New Brunswick

For essential background and terminology I refer to [2] (of course), and to [5] and [3]. I will attempt to address some, but surely not all, of the following problems. You will find a few motivational remarks along the way.

1. Suppose $\mathcal{P}$ is any $(n-1)$-polytope. Give an accurate definition for the abstract pyramid $\mathcal{Q}$ with base $\mathcal{P}$. Compute $\Gamma(\mathcal{Q})$ in terms of $\Gamma(\mathcal{P})$. What happens when $\mathcal{P}$ is regular?

   Ditto for *bipyramid*, *prism*, etc. Gabe Cunningham has posed and looked into some of these problems.

   Remark: These operations are of course familiar and essential tools in convexity. Analogous operations such as suspension are crucial tools in algebraic topology.

   Remark: The monodromy group $\mathrm{Mon}(\mathcal{Q}) = \langle r_0, \ldots, r_{n-1}\rangle$ of an n-polytope $\mathcal{Q}$ is an sggi (string group generated by involutions). Each r_j simultaneously swaps all j-adjacent flag pairs, so that $\mathrm{Mon}(\mathcal{Q})$ acts as a permutation group on the flag set $\mathcal{F}(\mathcal{Q})$. The monodromy group $\mathrm{Mon}(\mathcal{Q})$ of an n-polytope $\mathcal{Q}$ says a lot about the regular covers of $\mathcal{Q}$. For example, when $\mathcal{Q}$ is regular, $\mathrm{Mon}(\mathcal{Q})$ and the automorphism group $\Gamma(\mathcal{Q})$ are isomorphic (as sggi's); see [3, Theorem 3.9].

2. Suppose the 3-polytope $\mathcal{Q}$ is a pyramid over a p-gonal base. What then is $\mathrm{Mon}(\mathcal{Q})$?

   Suppose $\mathcal{P}$ is any regular polytope of rank $(n-1)$. Can we say anything sensible about $\mathrm{Mon}(\mathcal{Q})$, when $\mathcal{Q}$ is the pyramid, bipyramid or prism over $\mathcal{P}$?

   Remark. A chiral polytope $\mathcal{P}$ and its enantiomorphic twin $\mathcal{P}^-$ are isomorphic yet different. They are distinguished intrinsically by standard 'rotations': if $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{n-1}\rangle$, then
   $\Gamma(\mathcal{P}^-) = \langle \sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \ldots, \sigma_{n-1}\rangle$. By mixing these groups we obtain $\Gamma(\mathcal{P})\diamondsuit\Gamma(\mathcal{P}^-)$. This subgroup of $\Gamma(\mathcal{P})\times\Gamma(\mathcal{P}^-)$ turns out to be isomorphic to the even subgroup $\mathrm{Mon}^+(\mathcal{P})$ of the monodromy group. Since $\mathrm{Mon}(P)$ therefore reconciles the left- and right-handed versions of $\mathcal{P}$, it was thought that $\mathrm{Mon}(\mathcal{P})$ would provide a common regular cover for $\mathcal{P}$ and $\mathcal{P}^-$. But

3. Rather unexpectedly, there is a chiral polytope $\mathcal{P}$ for which $\mathrm{Mon}(\mathcal{P})$ is not a string C-group. Is this a common phenomenon? (For such weird behaviour it is necessary that $\mathcal{P}$ not have regular facets or vertex-figures. See [3] for a lengthy discussion of such issues.)

4. The example I have in mind in (3) is the finite chiral 5-polytope of type $\{3,4,4,3\}$ recently described in [1]. The automorphism group $\Gamma(\mathcal{P})$ is $\mathbb{S}_6$, as generated by

$$\begin{array}{rcl}\sigma_1 & = & (1,2,3)\\ \sigma_2 & = & (1,3,2,4)\\ \sigma_3 & = & (1,5,4,3)\\ \sigma_4 & = & (1,2,3)(4,6,5).\end{array}$$

The middle section is the smallest chiral 3-polytope $\{4,4\}_{(2,1)}$.

Egon Schulte and I have proved that $\mathcal{P}$ does have a finite regular cover, with $2426112000000 = 2^{14}\cdot 3^6\cdot 5^6\cdot 13$ flags! Note that $\mathcal{P}$ itself has $1440 = 2\cdot 6!$ flags.

Classify the minimal regular covers of this $\mathcal{P}$.

5. Marston Conder has recently determined for each n the regular n-polytopes with minimal number of flags (and Schläfli symbol with no 2's):

`http://www.fields.utoronto.ca/programs/scientific/11-12/discretegeom/`
`gradcourses/RefMapsPolytopes-Lecture6.pdf`

Daniel Pellicer asked about this in Oaxaca, 2010. In [4, Problem 35] he asks the analogous question for chiral polytopes. A key problem seems to be that our repertoire of chiral polytopes is still very limited.

Are small chiral polytopes 'rare'? That is, can one usefully compare the number of isomorphism classes for regular versus chiral polytopes with a given rank n and having a given number g of flags?

References

[1] M. Conder, I. Hubard, and T. Pisanski, *Constructions for chiral polytopes*, J. London Math. Soc., 77 (2008), pp. 115–129.

[2] P. McMullen and E. Schulte, *Abstract Regular Polytopes*, vol. 92 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, UK, 2002.

[3] B. Monson, D. Pellicer, and G. Williams, *Mixing and monodromy of abstract polytopes.* in preparation.

[4] D. Pellicer, *Developments and open problems on chiral polytopes.* to appear.

[5] E. Schulte and A. I. Weiss, *Chiral polytopes*, in Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift, P. Gritzmann and B. Sturmfels, eds., vol. 4 of DIMACS Ser. Discrete Math. Theoret. Comput. Sci., Amer. Math. Soc., Assoc. Comput. Mach., 1991, pp. 493–516.