From Lecture 1

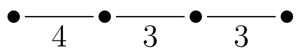
Notation: $V =$ a finite-dim'l vec. sp. over field $\mathbb{K}$; usually char. $p \neq 2$; dual space $= \check{V}$; identity $e \in GL(V)$. The pseudo-reflection $r = r_{\varphi,a} \in GL(V)$ is defined by $r(x) = x + \varphi(x)a$, for $x \in V$.

1. Show that $\det(r) = 1 + \varphi(a)$, so $\varphi(a) \neq -1$.
2. Suppose $r_{\varphi,a} \neq e \neq r_{\psi,b}$, where a, b are independent; then $r_{\varphi,a}$ and $r_{\psi,b}$ commute if and only if $\varphi(b) = 0 = \psi(a)$.
3. Suppose the pseudo-reflection $r_{\varphi,a}$ is an isometry for the non-singular orthogonal space $(V, \cdot)$. Then $r_{\varphi,a}$ must be a reflection (period 2), the root a must be non-isotropic (i.e. $a \cdot a \neq 0$) and
$$r_{\varphi,a}(x) = x - 2\,\frac{x \cdot a}{a \cdot a}\,a, \quad \forall x \in V. \tag{1}$$
Notation: write $r_a := r_{\varphi,a}$ or something similar in the case of ordinary reflections.
4. In our usual setup for a balanced reflection group, we have $J = \{0, \ldots, n-1\}$ and a basis $\{a_0, \ldots, a_{n-1}\}$ for V. Thus $G = \langle r_0, \ldots, r_{n-1} \rangle$, where $r_j = r_{\varphi_j, a_j}$ for various $\varphi_j \in \check{V}$.

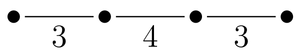
   Show that G acts irreducibly on V if and only if $\det(N) \neq 0$ and $\Delta(G)$ is connected.

From Lecture 2

1. The abstract Coxeter group B_4 has order $2^4\,4! = 384$ and diagram



   In fact, B_4 is a subgroup of index 3 in the abstract Coxeter group F_4 with diagram

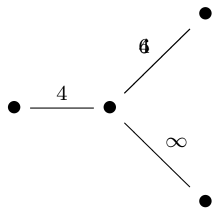


   Find a set of coset representatives (i.e. transversal) for B_4 as a subgroup of F_4.
2. (from [1]) Suppose that G is an irreducible subgroup of $GL(V)$ and is generated by reflections (of ordinary period 2). If G also leaves invariant a non-zero bilinear form $x \cdot y$, show that $x \cdot y$ must in fact be symmetric and non-singular.

From Lectures 3-4

1. (Pretend you don't know anything about the standard faithful representation $R : \Gamma \to G$.) Suppose the Coxeter group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ has only even branch labels (allowing ∞, and 2 for no branch at all). Use von Dyck's substitution theorem to prove that $|\Gamma| \geqslant 2^n$.

2. Determine the essentially distinct modified diagrams $\Delta(G)$ (representing invariant lattices) for the crystallographic Coxeter group with diagram $\Delta_c(G)$:



From Lectures 5-6

1. Prove that $\cos(\pi/5) = \tau/2$ and that $4\cos^2(\pi/10) = \tau + 2$.

References

[1] N. Bourbaki, *Groupes et Algébres de Lie, Chapitres IV-VI*, Hermann, Paris, 1968.