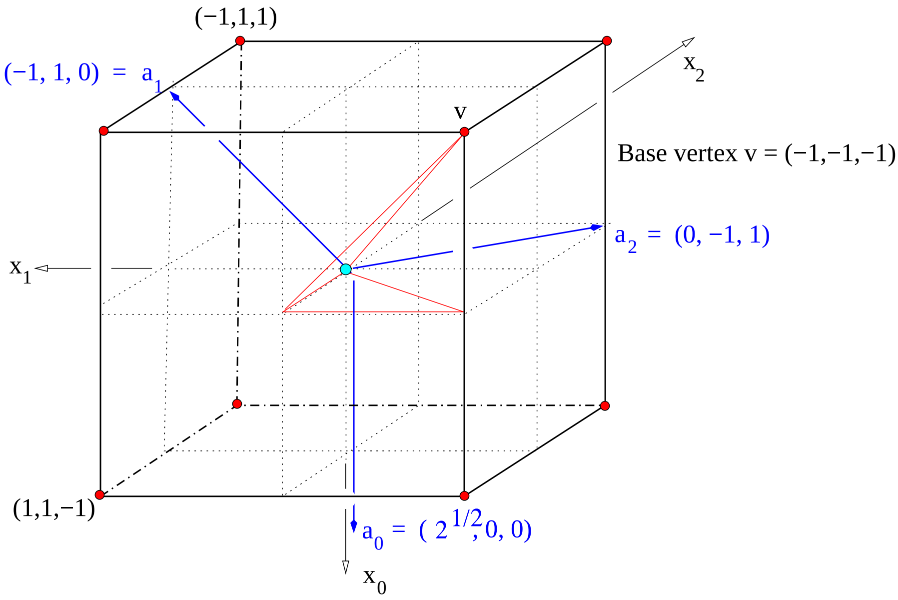
**Example**: the octahedral group $\Gamma = B_3$ with $\Delta_c = \bullet \overset{}{\underset{4}{—}} \bullet \underset{3}{—} \bullet$ is crystallographic. Here is the usual cube with vertices $(\pm1, \pm1, \pm1)$ and with rectangular axes conveniently out of the way:




To find a new basis vector $b_j$ just rescale the old $a_j$ (by $t_j > 0$) so that it first meets the given lattice $\Lambda$:

(a) The **cubic lattice** $\Lambda = \mathbb{Z}^3$ is $B_3$-invariant and returns as root lattice $Q = \Lambda$ itself: $b_0 = \frac{1}{\sqrt{2}}a_0 = (1, 0, 0)$; $b_1 = 1a_1$, $b_2 = 1a_2$. This information is included in the modified diagram

$$\Delta(B_3) = \overset{1}{\bullet} —— \overset{2}{\bullet} —— \overset{2}{\bullet}$$

(dependence on $\Lambda$ suppressed). We have labelled the nodes of $\Delta_c(B_3)$ by $t_0^2 = \frac{1}{2}$ , $t_1^2 = 1, t_2^2 = 1$, rescale the lot by 2 for convenience, then erase branch labels. We will see how to reconstruct $G$ from such a diagram.

(b) The **face-centred cubic lattice** $\Theta$ has integral vectors with even sum [4, 6D] and is also $B_3$-invariant. But rescaling is OK, so $\Lambda' = \frac{1}{\sqrt{2}}\Theta$ is also $G$-invariant and gives a new root lattice $Q'$ with modified diagram

$$\Delta(B_3)' = \overset{2}{\bullet} —— \overset{1}{\bullet} —— \overset{1}{\bullet}$$

(c) The **body-centred cubic lattice** consists of integral vectors with entries all even or all odd. But up to scale we merely get case (a) once again.

**Remarks**: $G$ may admit many essentially distinct invariant lattices. However, when the form $x \cdot y$ on $V$ is non-singular, and in particular when $G$ is finite, all $G$–invariant lattices can, in principle, be classified in a natural way ([1, 2, 3]).

# References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