Examples of Polygons, Polyhedra, Polytopes for Hand and Machine Calculation

1. The key reference is *Abstract Regular Polytopes* by Peter McMullen and Egon Schulte. See also the survey paper by McMullen.

   Coxeter in *Regular Polytopes* takes a more 'classical' approach, which is quite 'visual' and very rewarding. He is mainly concerned with the regular convex polytopes, which in *dimension*

   - $n = 2$ are the familiar regular convex polygons: $\{3\}$ = equilateral triangle; $\{4\}$ = square; $\{5\}$ = regular pentagon, ... In general. for any integer $p \geq 3$ we have a regular polygon $\{p\}$, which is unique up to similarity. In other words, for these classical examples, just being regular is enough to specify the shape up to rescaling.
   - $n = 3$ are the regular (or Platonic) solids $\{3, 3\}$, $\{4, 3\}$, $\{3, 4\}$, $\{3, 5\}$ and $\{5, 3\}$.

2. **Exercise**. Identify the regular polytopes of dimension $n = 1$; $n = 0$; $n = -1(???)$.

3. An *abstract regular n-polytope* $\mathcal{P}$ is a natural and wide-ranging generalization of the classical objects. Since $\mathcal{P}$ is a combinatorial object, actually a very special partially ordered set, we say $\mathcal{P}$ has *rank* n (rather than dimension n).

   To glimpse how this generalization comes about take an ordinary polyhedron like a cube; assemble its 8 vertices, 12 edges and 6 square facets into one set and declare that one of these 26 things is $\leq$ another when it lies on the other. From this you get a poset (partially ordered set), whose purely combinatorial properties ultimately suggest the definitions for their abstract counterparts.

Note that the rank n of the abstract polytope $\mathcal{P}$ is the number of <u>kinds</u> of *faces*, i.e. things from the list

vertex, edge, polygon,..., j-face, ..., $(n-1)$-face = facet.

Actually, to these *proper* faces we adjoin exactly two *improper* faces: F_n at the top (think the whole polytope), and F_{-1} at the bottom (think $\emptyset$).

Thus the abstract cube is a partially ordered set with a total of 28 elements.

Exercise. Sketch Hasse diagrams for the triangle $\{3\}$, the square $\{4\}$, and the cube $\{4,3\}$. Locate the first two diagrams as subdiagrams in the last.

4. Every abstract regular polytope $\mathcal{P}$ has a symmetery group G (also know as *automorphism group*), which has very special properties. Such groups are called *string C-groups*.

   A cornerstone of the theory is that, conversely, from each *string C-groups* G we can reconstruct an abstract regular polytope.

   Thus we can for now focus on understanding string C-groups, then later construct and investigate the corresponding regular polytope $\mathcal{P}$. We might write $\mathcal{P}(G)$ to indicate the polytope which we can reconstruct from G.

   Note that we often say 'regular polytope' when 'abstract regular polytope' is meant.

5. **Exercise**. Interpret the statements

   (a) $G(\mathcal{P})$

   (b) $\mathcal{P} \simeq \mathcal{P}(G(\mathcal{P}))$

   (c) $G \simeq G(\mathcal{P}(G))$

6. **Definition of string C-group**.

A *string C-group* G of rank n is a group with n *specified* generators having very special properties:

$$G = \langle r_0, \ldots, r_{n-1} \rangle \ .$$

A. Relations The generators *satisfy and fulfil* the following relations for specific integers $p_1, p_2, \ldots, p_{n-1}$ taken from $\{2, 3, \ldots, \infty\}$:

$$r_j^2 = 1, \text{ for } 0 \leq j \leq n-1$$

$$(r_{j-1} r_j)^{p_j} = 1 \text{ for } 1 \leq j \leq n-1$$

$$(r_i r_j)^2 = 1 \text{ whenever } |i - j| > 1$$

B. The Intersection Condition For any subsets $I, J \subseteq \{0, 1, \ldots n-1\}$ we have

$$G_I \cap G_J = G_{I \cap J} \ .$$

Note: by definition, $G_I := \langle r_k : k \in I \rangle$ is the subgroup of G generated by the r_k's as the subscript k runs through the indexing subset I.

This concludes the definition of string C-group.

7. **Comments**.

(a) The first set of equations in A demand that each generator r_j have period 2. In other words r_j is an involution. Intuitively, we think of r_j as a reflection, although on this combinatorial level, there are as yet no mirrors around.

(b) The second set of equations specifies that $r_{j-1} r_j$, which we might think of as a rotation, has period p_j. We allow $p_j = 2$; and $p_j = \infty$ merely says that $r_{j-1} r_j$ has infinite period, which means of course that the whole group G could not be finite. Another way to say this is that $r_{j-1} r_j$ has no equation to satisfy. Think of reflections in parallel mirrors, for which the product - a translation - has infinite period.

(c) The third set of equations says that r_0r_2, r_0r_3, r_1r_3, etc. all have period 2. But this means, more usefully, that r_0 commutes with each of r_2, r_3, etc.

(d) **Exercise**. If elements g and h in the group G each have period 2, prove that gh has period 2 if and only if $gh = hg$.

(e) The r_j's may well satisfy other relations which are 'independent' of those explicitly set out in A. This is OK, so long as the intersection condition B still holds.

(f) **Exercise**. Identify $n, p_1, \ldots, p_{n-1}$ for the regular octahedron; the regular pentagon; and the ordinary tiling of Euclidean space by unit cubes.

(g) **Exercise**. By definition, G_I, G_J and $G_{I \cap J}$ are certain *standard* subgroups of G. From our investigation of the cube, we know that they are crucial objects.

Check that

i. $G_I \cap G_J$ is also a subgroup of G.

ii. we always have

$$G_I \cap G_J \supseteq G_{I \cap J} \ .$$

I'll give an example with actual inequality; since the intersection condition B is then violated, such examples suggest geometric structures in which some features of a polyhedron say are destroyed.

(h) **Aside**. Certainly another set of generators for the same group G may fail to have these properties; then G is not a string C-group for those bad generators. On the other hand, the same group G could have yet another set of nice generators for which it becomes a new and different kind of string C-group.

8. **Definition**. If the regular n-polytope $\mathcal{P}$ has the above group G, we say that $\mathcal{P}$ has *Schläfli symbol*

$$\{p_1, p_2, \ldots, p_{n-1}\} \ .$$

9. **Exercise**. Give Schläfli symbols for the regular octahedron; for the bathroom tiling of the plane by identical hexagons; for the usual space filling by cubes.

   Sketch each of these regular polytopes. Locate a base flag and identify the generating reflections.

10. **Exercise**. Do the same as in the previous exercise for a new type of polytope. Take a 2×2 square (subdivided as usual into 4 squares) and glue opposite parallel edges together in the faniliar way to get a torus. What is n here? Identify the reflections. How big is G?

    Sketch a Hasse diagram.

11. **Exercise**. Write out the standard equations satisfied by the generators of the symmetry group G for a regular 4-polytope of type $\{p_1, p_2, p_3\}$.

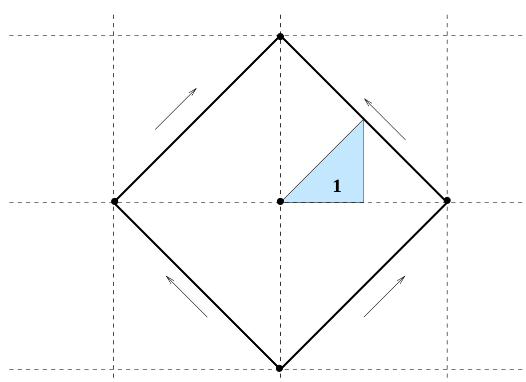
    We note that these generators might satisfy other equations not related to those in the standard list.

    In the worst case, how many subgroup checks are required to verify the intersection condition.

12. **Exercise**. Failure of the intersection condition.

    The usual tessellation of the plane by unit squares is the regular 3-polytope $\{4, 4\}$. Its edges are dotted in the figure below.

    Inscribed in the tiling is a single square with dark edges and area 2 (Check!). Also shaded in is a *base flag* (labelled 1) for the tiling $\{4, 4\}$.



(a) Locate generating reflections r_0, r_1, r_2 for the background $\{4, 4\}$ tessellation.

What is the order of group $G = \langle r_0, r_1, r_2 \rangle$? Write out the standard relations for these generators.

(b) Subdivide the dark square into copies of the base flag. You get say $q = ?$ copies.

(c) Now we create a torus by gluing together opposite sides of the dark square, as indicated by the arrows along its parallel sides. The torus inherits from the $\{4, 4\}$ tessellation various vertices, edges and square facets.

Compute the number v of distinct vertices on the torus, the number e of edges and the number f of squares.

Compute the *Euler characteristic* $\chi := v - e + f$ for this torus.

(d) Try to sketch this torus, donut style, with its vertices, (now curved) edges and (curved) squares.

(e) Note: The torus also inherits the q copies of the base flag and the reflections r_0, r_1, r_2. Strictly speaking we should call them by new letters, say $\tilde{r}_0$, etc., since we have changed the underlying geometry. But for simplicity let's stick with r_0, r_1, r_2.

With this new meaning, r_0, r_1, r_2 satisfy the old relations from before, plus new relations which correpond to the geometrical process of gluing together the opposite edges.

(f) Label the q copies of the base flag $1, \ldots, q$. (You already have 1 labelled.) The new r_j's thus act as permutations on $\{1, 2, \ldots, q\}$. Write out these permutations of degree q.

Enter them on Gap and construct the corresponding group G. What is the order of G? Should you be surprised?

(g) Check that the standard relations for the new generators *continue to hold*. However, as I said, some new relations must also hold. To see what, let's look at the geometry.

Look at the the translation t which runs diagonally $\sqrt{2}$ units in the northeast direction. It is exactly this translation which has the effect of gluing the southwest edge of the dark square to the northeast edge. Of course, in the orginal $\{4, 4\}$ tessellation, the translation t has infinite period. (And $\{4,4\}$ is an infinite polyhedron; see part (a) above.)

Now t is a symmetry of the tessellation $\{4,4\}$. Since the original r_j's generate the infinite symmetry group for $\{4,4\}$, we must be able to express t in terms of r_0, r_1, r_2. Do that. (Hint: use refactoring; since t is direct, you should obtain a word of even length in the r_j's.)

(h) Now, from the point of view of a bug living on the torus the 'new' t is the identity! Check that in the new finite group G the generators r_j satisfy an extra equation corresponding to the way that t factors.

(i) Now look at the intersection condition on this finite group. We shall see that the crucial check concerns $I = \{0,1\}$ and $J = \{1,2\}$, so that $I \cap J = \{1\}$. Then $G_I \cap G_J \stackrel{?}{=} G_{I \cap J}$ becomes

$$\langle r_0, r_1 \rangle \cap \langle r_1, r_2 \rangle \stackrel{?}{=} \langle r_1 \rangle \ .$$

Check what happens for the current group G.

(j) Sketch the Hasse diagram for this structure. Since the intersection condition fails in part, this structure is not a 3-polytope (polyhedron), although it does describe a perfectly good torus. Can you see in the diagram where 'polytopality' fails geometrically?

13. **Exercise**(still with the tessellation $\{4,4\}$). If the identifying translation t has the vector $(b,0)$, for some integer $b \geq 2$, then we do actually get regular 3-polytope, or *regular toroid* for short. (When $b = 1$ there is a structural flaw similar to that in the previous question.)

We denote this toroid by $\{4,4\}_{(b,0)}$. For it, find v, e, f and the order of G. Write down a full set of relations, both standard and special, for this group.

14. Let's leave the torus and move to the sphere. A rubbery cube $\{4, 3\}$ inflates to a sphere, so we might say that the corresponding group $G = \langle r_0, r_1, r_2 \rangle$ is of *spherical type*. We saw that the standard relations were

$$r_0^2 = r_1^2 = r_2^2 = (r_0 r_1)^4 = (r_1 r_2)^3 = (r_0 r_2)^2 = 1\ . \tag{1}$$

Remarks: In fact, these are the only defining relations needed for the group G, which we recall has order 48. Accordingly, G is an example of a *Coxeter group*.

On my summer-stuff website you can find a Gap file "refgp" for dealing with such relations and the groups which they define.

Recall that the *central isometry* z sends each vertex of the cube to its antipode. As a matrix, z corresponds to $-I_{3\times 3}$.

Here we are going to play the same trick with the central inversion z as we did with the translation t on the square tessellation.

Exercises.

(a) Represent G on Gap by permutations of the vertices labelled $1, \ldots, 8$.

(b) Just looking at your cube, write down z as a permutation.

(c) What does it mean to say that z is a central element in G?

(d) Show that the subgroup $K = \langle z \rangle$ is the centre of G. Use Gap if you wish. What is the order of K?

(e) Show that $K \triangleleft G$ (K is normal in G).

(f) Use Gap, or otherwise, construct the quotient group $Q := G/K$. The natural homomorphism $\varphi : G \to K$ is onto, so that the r_j's map to generators $s_j := \varphi(r_j)$ of the quotient group.
So define s_j this way in Gap.
Hint: if you define Q as above, then Gap will likely automatically produce what you want. The first generator s_0 will then be Q.1 in Gap.

(g) Write out the standard relations satisfied by s_0, s_1, s_2 and check whether they really do hold, either by eye or on Gap.

(h) Trickier: write out the special extra relation satisfied by s_0, s_1, s_2 in Q.
Hint: taking the quotient amounts to making z equal the identity, now on a new space obtained from the sphere by identifying (=gluing together) all pairs of antipodal points. This new surface is called a *projective plane*. Like the Möbius band it is *non-orientable*.

Thus, back on the cube or on the sphere, you should factor the rotatory reflection z in terms of the reflections r_0, r_1, r_2. Since z is opposite, the resulting word in the r_j's will have an odd number of terms.

(i) Check on Gap that the corresponding word in the s_j's gives the identity. From the point of view of a bug on the projective plane, z equals 1.

(j) Check the intersection property for $Q = \langle s_0, s_1, s_2 \rangle$.
Hint: we must check that

$$Q_I \cap Q_J = Q_{I \cap J}$$

for all $I, J \subseteq \{0, 1, 2\}$.

At the worst, there are $8^2 = 64$ choices for the two subsets I and J of $\{0, 1, 2\}$. Well, since the test is symmetric in I and J, there are really only 32 tests. And, in fact, several of the tests work automatically; for example, if either I or J equals the whole set $\{0, 1, 2\}$ or the empty set $\emptyset$, then $Q_I \cap Q_J = Q_{I \cap J}$ simply becomes $Q_J = Q_J$, which is true anyway. In other words, we can ignore several extreme cases. Note that $Q_\emptyset = \{1\}$, being the smallest group containing nothing.

Anyway, a more careful analysis shows that in rank 3 we actually need to perform only two checks:

- that each s_j really does have period 2; and
- that $Q_{\{0,1\}} \cap Q_{\{1,2\}} = Q_{\{1\}}$

Check these conditions on Gap, or otherwise, for the group Q.

(k) Note: in the non-polyhedral torus a few questions back, you should have found that this last equality failed! (We had r_j generating G there instead of s_j generating Q.)

(l) By now you should know the order of the rank 3 string C-group Q. From it, we get an abstract regular polyhedron $\mathcal{H}$, for *hemi-cube.* Compute v, e, f for it. Now compute the Euler characteristic of the projective plane.

We will denote the hemi-cube by $\{4, 3\}_3$. Can you guess why?

Try to sketch $\mathcal{H}$ in a way that is reminiscent of our earlier diagram for the torus.

(m) What group is Q exactly? What more familiar 3-dimensional structure has symmetry group Q? Can we use this fact to create an accessible model for the unfamiliar polyhedron $\mathcal{H}$?

15. Repeat the key parts of the previous question for another centrally symmetric Platonic solid, the regular icosahedron $\{3, 5\}$ with 12 vertices. This will be crucial to our summer project.

Exercises

(a) We begin with the group $G = \langle r_0, r_1, r_2 \rangle$ for the icosahedron. What is its order? You may want to get it on Gap, say by permutations of the 12 vertices.

(b) Factor the central element z in terms of the r_j's.

(c) Compute the quotient group $Q := G/\langle z \rangle$. What is its order?

(d) Identify Q and check the intersection condition, as in 14(j).

(e) From Q we get a regular polyhedron called the 'hemi-icosahedron' and denoted $\{3, 5\}_5$. Compute v, e, f and the Euler characteristic.

(f) Where did the subscript 5 come from?

(g) From the same group Q we get the dual polyhedron, namely the 'hemi-dodecahedron' $\{5, 3\}_5$. There is no need to recompute: we simply relabel the old generator s_0 for Q as s_2 and the old s_2 as s_0. The subgroups Q_0, etc. follow suit.

Put even more simply, just take the Hasse diagram for $\{3, 5\}_5$ and turn it upside down to get the Hasse diagram for $\{5, 3\}_5$!

Our 4-polytope $\{\{3,5\}_5\,,\,\{5,3\}_5\}$

16. We need some background on finite fields and matrix groups over them.

    The set
    $$\mathbb{Z}_{11} = \{0,1,2,3,4,5,6,7,8,9,10\} \qquad (2)$$
    becomes a *field* when we work with its elements modulo 11. (We get a field since 11 is prime.)

    Thus we can add, subtract, multiply and divide (except by 0), according to the 'usual rules' of arithmetic, though with unconventional results, of course. Since all operations are to be closed, every end computation must be one of the 11 numbers in (2) above.

    For example, as ordinary integers we have $5 \cdot 9 = 45 \equiv 1 \pmod{11}$. Thus in $\mathbb{Z}_{11}$ we have $5 \cdot 9 = 1$, so that
    $$\frac{1}{5} = 9 \text{ and } \frac{1}{9} = 5\ .$$

    Exercises

    (a) Compute $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{10}$.

    (b) Compute $-1, -3, \frac{-4}{7}, \frac{9}{2} + \frac{3}{5}$.

    (c) Determine $\{x^2 : x \in \mathbb{Z}_{11}\}$.

    (d) Find all solutions, if any, to the following equations:
    $$x^2 = 1\ ,\ x^2 = -2\ ,\ x^2 + 8 = 0\ ,\ x^2 = 5$$
    $$x^2 + x + 2 = 0$$

17. Matrix algebra, including determinants, inverses for square matrices, row reduction, basis theorems, etc. works in the customary way over any field, in particular over $\mathbb{Z}_{11}$. Consider this matrix with entries from $\mathbb{Z}_{11}$:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

Then $\det(A) = 2 \cdot 5 - 3 \cdot 4 = 9 \neq 0$, so that

$$A^{-1} = \frac{1}{9} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = 5 \begin{bmatrix} 5 & 8 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 2 & 10 \end{bmatrix} .$$

Check this!

18. **Exercise**. Verify directly for general 2×2 matrices

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \quad , \quad A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

that

$$\det(A_1 A_2) = \det(A_1)\det(A_2) \ . \tag{3}$$

Since your calculation uses only ordinary algebra, it will work over any field, in particular over $\mathbb{Z}_{11}$.

19. **Exercise**. How many 2×2 matrices over $\mathbb{Z}_{11}$ are there?

20. **Exercise**. Let

$$V = \mathbb{Z}_{11}^2 = \mathbb{Z}_{11} \times \mathbb{Z}_{11} = \{(x, y) : x, y \in \mathbb{Z}_{11}\} \ .$$

How many points are there in V?

With the usual componentwise operations on the row vectors (x, y), V becomes a 2-dimensional vector space over $\mathbb{Z}_{11}$. The standard basis is just

$$e_1 = (1, 0), e_2 = (0, 1) \ .$$

I am using rows here for convenient type-setting; but columns work just as well, and are more suitable for right to left calculation, as we have seen.

21. **Exercise**. The *line* spanned by any fixed non-zero vector $(a, b) \in V$ is

$$L := \{t(a, b) : t \in \mathbb{Z}_{11}\} .$$

This line is a 1-dimensional subspace of V.

How many points lie on such a line?

22. **Exercise**. Recall that a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if its rows are linearly independent, that is, if and only if neither row is a multiple of the other.

Since $(0, 0) = 0(c, d)$, this means that A^{-1} exists if and only if the top row (a, b) is non-zero and the bottom row (c, d) is <u>not</u> on the the line L spanned by the top row.

Using these ideas, how many invertible 2×2 matrices A are there over $\mathbb{Z}_{11}$?

23. **Definition**. The *general linear group*

$$GL(2, 11) := \{2 \times 2 \text{ invertible matrices } A \text{ over } \mathbb{Z}_{11}\} .$$

Verify that $GL(2, 11)$ really is a group with matrix multiplication as the operation. You have to check that A, B invertible implies that A^{-1} and AB are themselves invertible. This is almost obvious! Or you could use (3).

24. **Exercise**. What is the order of $GL(2, 11)$?

25. **Exercise**. Look at the set of 10 non-zero elements in $\mathbb{Z}_{11}$, namely

$$\mathbb{Z}_{11}^* := \{x \in \mathbb{Z}_{11} : x \neq 0\} .$$

Check that $\mathbb{Z}_{11}^*$ is a group (of order 10) under multiplication modulo 11. In fact, check that it is a cyclic group.

26. **Exercise**. Define a function

$$\begin{array}{rcl} \delta : GL(2,11) & \to & \mathbb{Z}_{11}^* \\ A & \mapsto & \det(A) \end{array}$$

What equation guarantees that δ a group homomorphism?

Prove that δ is onto.

What is the order of the quotient $GL(2,11)/\ker(\delta)$?

27. **Definition**. The *special linear group*

$$SL(2,11) := \ker(\delta) \ .$$

What is the order of $SL(2,11)$?

Simply put, $SL(2,11)$ is the group of all 2×2 matrices over $\mathbb{Z}_{11}$ with determinant 1.

28. **Exercise**. By hand, compute the center of $SL(2,11)$. Check on Gap.

Hint: a central element is a 2×2 matrix

$$Z = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

which has determinat $xt - yz = 1$ and which commutes will *all* matrices of determinant 1. So see what this means for simple matrices like

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} .$$

29. **Exercise**. The *center* Cen of $SL(2,11)$ consists of all such matrices Z. Show that Cen is normal subgroup of $SL(2,11)$ and determine its order.

30. **Definition**. The *projective special linear group*

$$PSL(2,11) := SL(2,11)/Cen \ .$$

So we have a quotient group. What is its order?

31. **Calculating in** $PSL(2,11)$.

You should have found that $Cen = \{\pm I\}$, where I is the 2×2 identity matrix. Thus each individual element in the quotient group $PSL(2,11)$ is really a coset, typically $A \cdot Cen = A\{\pm I\} = \{\pm A\}$. Such a coset is to be considered as one entity on the quotient group. Intuitively, we 'identify' matrix A and its negative $-A$ (both of which have determinant 1).

In practical terms, we just multiply 2×2 matrices of determinant 1 over $\mathbb{Z}_{11}$ as usual, but think of A and $-A$ as being the 'same'.

32. **Example**. Let

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ,$$

(if our field were the reals we would think 90° rotation here). Now

$$R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I ,$$

which generates the center Cen. Thus, as an element $r := R \cdot Cen = \{\pm R\} \in PSL(2,11)$, we simply have

$$r^2 = 1 .$$

We shall soon think of such r's as 'reflections'.

33. **Project**. Find four generators r_0, r_1, r_2, r_3 for PSL(2,11), which satisfy the standard relations for a certain 4-polytope, namely

$$r_0^2 = r_1^2 = r_2^2 = r_3^2 = (r_0 r_1)^3 = (r_1 r_2)^5 = (r_2 r_3)^3 = 1 ,$$

$$(r_0 r_2)^2 = (r_0 r_3)^2 = (r_1 r_3)^2 = 1 ,$$

along with other special relations which embody the projective plane nature of the facets and vertex figures:

$$(r_0 r_1 r_2)^5 = (r_1 r_2 r_3)^5 = 1 .$$

Hint: on Gap one can manufacture the matrix group $SL(2,11)$ along with a 2 : 1 homomorphism f from $SL(2,11)$ to some permutation group, which is easier to work with. Then try to find four generating permutations which satisfy the equations and work backward, using f^{-1} to find the matrices. Of course, f^{-1} will then produce a matrix and its negative.

Or you can work by hand. Pick a convenient matrix for r_0 to start. Whenever you need something equal 1, the corresponding matrix product can be off by the negative. For example, on the level of matrices, either $R_0^2 = -I$ or $R_0^2 = I$ will be OK.

The easiest place to start is with the commuting relations. For example, say you know R_0, but not R_2, which we therefore write as an unknown matrix

$$R_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ,$$

of determinant $1 = ad - bc$. Then choose to work with either $R_0 R_2 = R_2 R_0$ or $R_0 R_2 = -R_2 R_0$; this gives some equations, which you can use to simplify R_2. A good bit of inspired guesswork, or brute force trial and error on Gap, may be needed.

34. **Exercise**. Check that $G = \langle r_0, r_1, r_2, r_3 \rangle$ satisfies the intersection condition, so that we really do have the symmetry group of a 4-polytope $\mathcal{P}$. This polytope will be denoted

$$\{\{3, 5\}_5 \,,\, \{5, 3\}_5\} \,.$$

How many vertices, edges, polygons (= 2-faces), and facets (= 3-faces) does $\mathcal{P}$ have?

35. The v vertices correspond to right cosets of the subgroup $G_0 := \langle r_1, r_2, r_3 \rangle$. This will give a very convenient permuatation representation of degree v for the group G.

Get the r_j's in this form. (Quite possibly you already have this.)

Now every permutation of degree v amounts to a $v \times v$ permutation matrix over the reals $\mathbb{R}$. (See your linear algebra text.) Get these matrices; there are built in commands in Gap for this. But to avoid confusion with the 2×2 matrices mod 11, call these new real matrices something else, say p_0, p_1, p_2, p_3. Check that the resulting matrix group has the correct order.

Now you can begin to see why we have to work with such a high-dimensional real model for the polytope $\mathcal{P}$.