Introduction: Regular convex polytopes certainly have 'reflection symmetry'. We often think the same way when imagining abstract regular polytopes. But what are reflections and how might they generate the automorphism group $\Gamma(\mathcal{P})$ of an abstract polytope $\mathcal{P}$? We first take a brief look at abstract things. See Section 1 of the notes posted at

`http://www.math.unb.ca/~barry/fields/outline.pdf`

Abstract Polyhedra: If the *abstract regular* 3*-polytope* (or polyhedron) $\mathcal{P}$ has type $\{p_1, p_2\}$, its full automorphism group $\Gamma(\mathcal{P}) = \langle\rho_0, \rho_1, \rho_2\rangle$ satisfies (at least!) relations $\rho_j^2 = (\rho_0\rho_1)^{p_1} = (\rho_1\rho_2)^{p_2} = (\rho_0\rho_2)^2 = 1$, as well as the *intersection condition*

$$\langle\rho_0, \rho_1\rangle \cap \langle\rho_1, \rho_2\rangle = \{1\}. \tag{1}$$

You may think of such a polyhedron $\mathcal{P}$ as a (fully) regular map, made *polyhedral* by (1).

Typical fonts: Greek letters as above in 'abstract' situations, or for finitely presented groups; Roman letters, as in $G = \langle r_0, \ldots, r_{n-1}\rangle$, for a 'concrete' linear group, where the r_j's are actual reflections.

Reflection Groups (Section 2 of notes)
Notation: V = a finite-dim'l vec. sp. over field $\mathbb{K}$; usually char. $p \neq 2$; dual space = $\check{V}$; identity $e \in GL(V)$.

Definition. $r \in GL(V)$ is a *pseudo-reflection* if rank$(r - e) = 1$. Thus, there exist a vector $a \in V$ and linear map $\varphi \in \check{V}$, both non-zero, such that

$$r(x) = x + \varphi(x)a, \quad \forall x \in V. \tag{2}$$

Sometimes: write $r_{\varphi,a}$ for r to emphasize details; allow $r_{0,a} = r_{\varphi,0} = e$.

Exercises. (a) Show that $\det(r) = 1 + \varphi(a)$, so $\varphi(a) \neq -1$.
(b) Show that the *fixed space* V^r equals $\ker\varphi$ (which has codimension 1 in V).
(c) Observe that the *direction space* $V_r := \{r(x)-x \,:\, x \in V\}$ equals $\mathbb{K}a$, which is 1-dimensional.
(d) What possible periods can a pseudo-reflection have when $\mathbb{K} = GF(p^t)$? When $\mathbb{K} = \mathbb{C}$?
(e) Suppose $r_{\varphi,a} \neq e \neq r_{\psi,b}$, where a, b are independent; then $r_{\varphi,a}$ and $r_{\psi,b}$ commute if and only if $\varphi(b) = 0 = \psi(a)$.

Thus $V_r \subseteq V^r$ if–if $\varphi(a) = 0$, in which case r is defined to be a *transvection*.

Exercise. A transvection r has period p, if $p > 2$, or period ∞, when $p = 0$.

(True) Reflections: If the pseudo-reflection r is not a transvection, then $V = V^r \oplus V_r$, and r acts as the scalar $1 + \varphi(a)$ on V_r. Since $p \neq 2$, we conclude that r is involutory if and only if $\varphi(a) = -2$. Because r then acts as -1 on V_r, we call r a *reflection* (in the usual sense). In this case, we say that a is a *root* for r and that $V^r = \ker\varphi$ is the *mirror* for r.

Exercises on Reflections. If $r_{\varphi,a},\ r_{\psi,b}$ are reflections, and $\varphi(b) = 0 \neq \psi(a)$, then the commutator $[r_{\varphi,a},\ r_{\psi,b}] = (r_{\varphi,a}r_{\psi,b})^2$ is the non-trivial transvection $r_{\varphi,c}$, where the new root $c = \psi(b)a$.
Pathology: In such cases, distinct reflections have the same mirror $\ker\varphi$. But orthodoxy is enforced by orthogoxy:

Orthogonal Geometries and Their Reflections. An excellent source is [1].

Later V will often come equipped with a *symmetric bilinear form* $x \cdot y$, whose *radical* (subspace)

$$\mathrm{rad}V := \{x \in V \mid x \cdot y = 0,\ \forall y \in V\}\ .$$

The *orthogonal space* V is *non-singular* if $\mathrm{rad}V = \{o\}$. A vector x is *isotropic* if $x \cdot x = 0$.

The subgroup of $GL(V)$ which leaves $x \cdot y$ invariant is the *orthogonal group* $O(V)$ corresponding to this form.

Exercises. (a) If V is non-singular and $g \in O(V)$, then $\det(g) = \pm 1$.
(b) Suppose the pseudo-reflection $r_{\varphi,a}$ is an isometry for the non-singular orthogonal space $(V, \cdot)$. Then $r_{\varphi,a}$ must be a reflection (period 2), the root a must be non-isotropic and

$$r_{\varphi,a}(x) = x - 2\,\frac{x \cdot a}{a \cdot a}\,a,\ \ \forall x \in V. \tag{3}$$

Notation: we often write $r_a := r_{\varphi,a}$ in this case.
(c) Do the rest of Lemma 2.1 in the notes.

Remark: Pseudo-reflections of period $q > 2$, such as occur in some of the unitary groups described in [3] or [2], will not concern us here. But we must confront unitary reflections of period 2.

References

[1] E. Artin, *Geometric Algebra*, Interscience, New York, 1957.

[2] A. M. Cohen, *Finite complex reflection groups*, Ann. Sci. Ecole Norm. Sup. (4), 9 (1976), pp. 379–436.

[3] H. S. M. Coxeter, *Regular Complex Polytopes*, Cambridge University Press, Cambridge, UK, 2nd ed., 1991.