Seeking Balance: If $r_{\varphi,a}$, $r_{\psi,b}$ are reflections, and $\varphi(b) = 0 \neq \psi(a)$, then $r_{\psi,b}r_{\varphi,a}r_{\psi,b} = r_{\varphi,c}$, where $c = a + \psi(a)b$. Thus two (conjugate) reflections $r_{\varphi,a}$ and $r_{\varphi,c}$ share the same mirror but have independent roots (check this peculiarity).

Main Concern: a group $G \subseteq GL(V)$ generated by a finite set of reflections, typically

$$G = \langle r_j \,|\, j \in J \rangle, \text{ with } r_j = r_{\varphi_j, a_j}, \text{ and } \varphi(a_j) = -2 \,.$$

Definition: the *reflection group* G, with the specified generators r_j, is *balanced* if $\varphi_j(a_k) = 0$ implies $\varphi_k(a_j) = 0$ for $j, k \in J$.
Examples: when G contains no transvections, in particular, if G is a group of isometries for some non-singular orthogonal space V.

Definition. For G balanced the graph $\Delta(G)$ has node set J, distinct $j, k \in J$ are adjacent whenever $\varphi_j(a_k) \neq 0$. The $|J| \times |J|$ matrix $N := [\varphi_i(a_j)]$ is called a *Cartan matrix* for G (with respect to the specified generating reflections; see [3, §1] for implications when $\mathbb{K} = \mathbb{R}$).
Remark: By attaching labels to nodes or branches we get Coxeter diagrams Δ_c and their various kin.

Our Standard Setup: $J = \{0, \ldots, n-1\}$ and $\{a_0, \ldots, a_{n-1}\}$ is a basis for V.

Exercise (do for sure!) If, as above, $\{a_0, \ldots, a_{n-1}\}$ is a basis for V, then G acts irreducibly on V if and only if $\det(N) \neq 0$ and $\Delta(G)$ is connected.

Coxeter Groups. An *abstract Coxeter group* Γ is defined through a special sort of presentation:

$$\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \,:\, (\rho_i\rho_j)^{p_{ij}} = 1, \quad 0 \leqslant i, j \leqslant n-1 \rangle, \tag{1}$$

where $p_{ii} = 1$ and $2 \leqslant p_{ij} = p_{ji} \leqslant \infty$ for all $i \neq j$. A *string Coxeter group* has $p_{ij} = 2$, equivalently ρ_i commuting with ρ_j, whenever $|i - j| \geqslant 2$.

The standard real representation for the Coxeter group Γ: On *real* n-space V, with basis $a_0, \ldots, a_{n-1}$, define a symmetric bilinear form $x \cdot y$ by setting

$$a_i \cdot a_j := -2\cos\frac{\pi}{p_{ij}}, \quad 0 \leqslant i, j \leqslant n-1.$$

Thus $a_j^2 := a_j \cdot a_j = 2$ and $r_j(x) = x - (x \cdot a_j)a_j$ is an isometric reflection on V.

Theorem (Bourbaki, i.e. Tits?) [2, Ch. 5.3-5.4] The mapping $\rho_j \mapsto r_j$ induces a faithful representation

$$R : \Gamma \to G := \langle r_0, \ldots, r_{n-1} \rangle$$

of Γ in the orthogonal group $O(V)$.
We may put Γ aside and work instead with the *linear Coxeter group* G. (See [3, 4] for further properties of more general linear Coxeter groups.)

Of course, G is balanced and therefore has a diagram $\Delta(G)$, from which we obtain the familiar *Coxeter diagram* $\Delta_c(G)$ for G (and for Γ) as follows: whenever $p_{ij} \geqslant 3$ label the branch connecting nodes i, j by p_{ij}. (If $p_{ij} = 2$, nodes i, j are non-adjacent. The very common label 3 is often suppressed.)

Example: dihedral groups $I_2(q)$ of order $2q$ – symmetry group for q-gon $\{q\}$

Example: group B_n of order $2^n n!$ for the n-cube $\{4, 3, \ldots, 3\}$

Spherical Cases: G ($\simeq \Gamma$) is finite if–f when $x \cdot y$ is positive definite ([2, Th. 6.4]). We may call G an *orthogonal group generated by reflections* or even *spherical*, since it leaves invariant the unit sphere $\mathbb{S}^{n-1}$ in V.

Irreducible spherical (hence finite) G ([1]): A_n ($n \geqslant 1$), B_n ($n \geqslant 2$), D_n ($n \geqslant 4$), E_n ($n = 6, 7, 8$), F_4, H_3, H_4, or $I_2(q)$ (dihedral of order $2q$).
See Table for Coxeter diagrams; convenient to use same label for group, diagram, root system, depending on your focus.

Euclidean Cases: $x \cdot y$ positive semidefinite, with $\dim(\mathrm{rad} V) = 1$; G (infinite) is of *Euclidean type*; acts on affine Euclidean $(n-1)$-space $\mathbb{E}^{n-1}$ [2, ch. 4].

Hyperbolic Cases: $x \cdot y$ is non-singular with signature $(+ + \ldots + -)$; G (infinite) is of *hyperbolic type*; acts on hyperbolic $(n-1)$-space $\mathbb{H}^{n-1}$ [2, §6.8-6.9].

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