Recap: The general abstract Coxeter group with n generators is

$$\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \,:\, (\rho_i \rho_j)^{p_{ij}} = 1, \quad 0 \leqslant i, j \leqslant n-1 \rangle,$$

where $p_{ii} = 1$ and $2 \leqslant p_{ij} = p_{ji} \leqslant \infty$ for all $i \neq j$. The representation $R : \Gamma \to G$ is faithful. Recall V has real basis $a_0, \ldots, a_{n-1}$ and symmetric bilinear form $x \cdot y$ specified by setting $a_i \cdot a_j := -2\cos \pi / p_{ij}$. Then ρ_j can be identified with the isometric reflection r_j, where $r_j(x) = x - (x \cdot a_j)a_j$. Thus $G = \langle r_0, \ldots, r_{n-1} \rangle$ is a subgroup of $O(V)$.

Definition. Γ is *crystallographic* (w.r.t. R) if G leaves invariant some *lattice* Λ (i.e. $\mathbb{Z}$-module spanned by some basis of V.)

Now what? As in [4, §1] look at (the subgroup!) $\Lambda_i := \Lambda \cap \mathbb{R}a_i$. Then $\Lambda_i = \mathbb{Z}b_i$, where $b_i = t_i a_i$, for some real $t_i \geqslant 0$. But all $t_i > 0$ and $\{b_0, \ldots, b_{n-1}\}$ is a (new) *root basis* for V.

The *root lattice* $Q := \sum_{i=0}^{n-1} \mathbb{Z}b_i$ is a G-invariant sublattice of Λ. (Suppress dependence on Λ.) In fact,

$$r_i(b_j) = b_j + m_{ij} b_i$$

for *Cartan integers* $m_{ij} = -t_j(a_i \cdot a_j)/t_i$. Furthermore, we can interpret G, hence also the Coxeter group Γ, as a subgroup of $GL_n(\mathbb{Z})$ (= invertible $n \times n$ integral matrices). Modulo $p > 0$ we get a finite group G^p.

Exercises: (a) Check that $[m_{ij}]$ is the Cartan matrix for this new representation of Γ.

(b) Check that r_i and r_j $(i \neq j)$ commute if–f $p_{ij} = 2$ if–f $m_{ij} = m_{ji} = 0$.

Example: the "usual" octahedral group $\Gamma = B_3$ with $\Delta_c = \bullet \underset{4}{\text{———}} \bullet \underset{3}{\text{———}} \bullet$.

(a) The invariant lattice $\Lambda = \mathbb{Z}^3$ returns $Q = \Lambda$ and modified diagram $\Delta = \overset{1}{\bullet} \text{ —— } \overset{2}{\bullet} \text{ —— } \overset{2}{\bullet}$.

(b) Let Θ be the face-centred cubic lattice (integral vectors with even sum [5, 6D]). But rescaling is OK! $\Lambda' = \frac{1}{\sqrt{2}}\Theta$ is also G-invariant and gives a new root lattice Q' with modified diagram $\Delta' = \overset{2}{\bullet} \text{ —— } \overset{1}{\bullet} \text{ —— } \overset{1}{\bullet}$.

Remarks: G may admit many essentially distinct invariant lattices. However, when the form $x \cdot y$ on V is non-singular, and in particular when G is finite, all G–invariant lattices can, in principle, be classified in a natural way ([1, 3, 4]).

The general crystallographic Coxter group Γ: All $m_{ii} = -2$; and $m_{ij} \geqslant 0$ for $i \neq j$. We can arrange $m_{ij} \leqslant m_{ji}$ in

$$m_{ij} m_{ji} = (a_i \cdot a_j)^2 = 4\cos^2(\pi/p_{ij}) \in \mathbb{Z}.$$

Data on dihedral subgroups $\langle r_i, r_j \rangle$

m_{ij}	m_{ji}	p_{ij}	$t_i/t_j = \sqrt{m_{ji}/m_{ij}}$	Subgraph on nodes i, j	Invar. quad. form
0	0	2	undet'd	$\overset{s}{\bullet}\quad\overset{t}{\bullet}$	$sx_i^2 + tx_j^2$
1	1	3	1	$\overset{s}{\bullet}\!-\!\!-\!\overset{s}{\bullet}$	$s(x_i^2 - x_i x_j + x_j^2)$
1	2	4	$\sqrt{2}$	$\overset{2s}{\bullet}\!-\!\!-\!\overset{s}{\bullet}$	$s(2x_i^2 - 2x_i x_j + x_j^2)$
1	3	6	$\sqrt{3}$	$\overset{3s}{\bullet}\!-\!\!-\!\overset{s}{\bullet}$	$s(3x_i^2 - 3x_i x_j + x_j^2)$
1	4	∞	2	$\overset{4s}{\bullet}\!-\!\!-\!\overset{s}{\bullet}$	$s(4x_i^2 - 4x_i x_j + x_j^2)$
2	2	∞	1	$\overset{s}{\bullet}\!=\!\!=\!\overset{s}{\bullet}$	$s(x_i - x_j)^2$

Note: s, t positive integers. We will mainly ignore the last line, which has minor uses.

Corollary - crystallographic restriction: If Γ is crystallographic, all basic rotation periods (=branch labels in Coxeter diagram $\Delta_c(\Gamma)$) must be $3, 4, 6$ or ∞ (or 2, for no branch).

The converse holds if $\Delta_c(\Gamma)$ is a *tree*, in particular a *string diagram.* See notes for full story, also Exercise 2 at top of page 48 in notes.

The new diagrams $\Delta(G)$ [6, 2]. The new representation of Γ as G in $GL_n(\mathbb{Z})$ can be reconstructed from the (new) Gram matrix $B = [b_i \cdot b_j]$, which in turn we encode in a new diagram $\Delta(G)$. Take the Coxeter diagram $\Delta_c(G)$ and label each node i by $b_i^2 = b_i \cdot b_i = 2t_i^2$. If nodes i, j connected by a branch labelled p_{ij}, then the ratio of larger to smaller node label is say $b_i^2/b_j^2 = 1, 2, 3$ or 4, as determined from the above chart. Now we can erase the branch labels. And on any connected component of $\Delta_c(G)$ we can rescale node labels so that such labels are positive integers s, t etc. (see the chart). Finally we can divide all such labels by their gcd.

Summary description of $\Delta(G)$: Node i is labelled b_i^2; and (a fortuitous artefact), for nodes $i \neq j$ joined by λ_{ij} branches, $b_i \cdot b_j = \frac{-\lambda_{ij}}{2}\max\{b_{ii}, b_{jj}\}$. From this it is easy to reconstruct the integral matrix representing r_i:

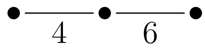
$$\begin{aligned} r_i(b_i) &= -b_i \\ r_i(b_j) &= b_j + \lambda_{ij}\max\{1, b_{jj}/b_{ii}\} b_i\ , \quad (i \neq j) \end{aligned}$$

The determinant of $B = [b_i \cdot b_j]$ equals the discriminant disc(V) of the orthogonal geometry V. This calculation is simplified if $\Delta(G)$ has a univalent node j, say adjacent to node k. If $B_{[j]}$ (resp. $B_{[j,k]}$) denotes the submatrix of B obtained by deleting row and column j (resp. j, k), then

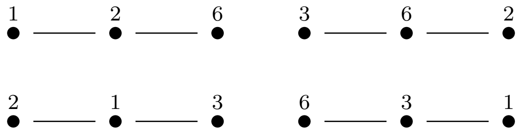
$$\det(B) = b_{jj}\det(B_{[j]}) - b_{jk}^2\det(B_{[j,k]}). \tag{1}$$

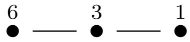
(Expand along row j [2, p.426].)

Example: the group G with Coxeter diagram



is crystallographic and acts naturally on $\mathbb{H}^2$. The essentially distinct root lattices (and representations of G in $GL_3(\mathbb{Z})$) are described by:



For example, the diagram  from above has Gram matrix

$$B = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 3 & -3/2 \\ 0 & -3/2 & 1 \end{bmatrix}$$

and associated quadratic form

$$f = 6x_0^2 - 6x_0x_1 + 3x_1^2 - 3x_1x_2 + x_2^2 \ .$$

Taking $j = 0, k = 1$, we note that $\det(B_{[0]}) = 3(1) - (-\frac{3}{2})^2 = \frac{3}{4}$, so that

$$\det(B) = 6(\frac{3}{4}) - (-3)^2(1) = -\frac{9}{2} \ . \tag{2}$$

The generating reflections are

$$r_0 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, r_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, r_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}.$$

But what on earth group do we get if we take these modulo the prime p?

References

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