Context: $\Gamma \simeq G = \langle r_0, \ldots, r_{n-1}\rangle$ is a crystallographic Coxeter group of rank n with string diagram, so $p_j = p_{j-1,j} \in \{2, 3, 4, 6, \infty\}$, for $1 \leqslant j \leqslant n-1$. We fix one of the node-labelled diagrams $\Delta(G)$ admitted by G, then represent G as a subgroup of $GL_n(\mathbb{Z})$. G can be finite (spherical) but most likely is not.

Choose an odd prime p. Thus $\mathbb{K} = \mathbb{Z}_p$ is a finite field.

Reduce G modulo p to get the finite group G^p. This can be done formally (via $G \otimes_{\mathbb{Z}} \mathbb{Z}_p$). But we work directly with matrices, abusing notation a bit: now $r_j \in Gl_n(\mathbb{Z}_p)$. Likewise, V is the space of column vectors over $\mathbb{Z}_p$, and $x \cdot y$, b_j, $B = [b_i \cdot b_j]$ etc. are interpreted naturally.

Observation 1. Changing one admissible diagram Δ to another Δ' is achieved by rescaling the b_j's by rationals of the form $2^x 3^y$, $(x, y \in \mathbb{Z})$. This is inconsequential if $p \geqslant 5$, but can be a pain when $p = 3$.

Convenient definition: If $p \geqslant 5$, or $p = 3$ but no branch of $\Delta_c(G)$ is marked 6, then we say that p is *generic* for G.

Observation 2. Each r_j remains a reflection mod p. Each finite period $p_i \in \{2, 3, 4, 6\}$ is unchanged mod p; but ∞ becomes p.

Mini-Theorem [1, Thm. 5.1]. In the rank 3 case, $G^p = \langle r_0, r_1, r_2\rangle$ is always a string C-group; hence we obtain an infinite family of regular polyhedra if G is infinite.

Proof. We saw most of this; non-generic cases can be checked in GAP. We haven't actually proved that the groups change as p changes! We haven't said what those groups are! $\square$

What's on sale? Finite reflection groups have been studied a lot. A complete census was finally achieved by Zalesskiĭ and Serežkin through very difficult papers [2, 3]. More specifically, suppose G is a finite, irreducible reflection group in $GL(V)$, where V has dimension $n \geqslant 3$ over field $\mathbb{K}$ (with characteristic $\neq 2$ and algebraic closure $\mathbb{L}$). Also suppose that G contains no non-trivial transvection. Then the key theorem in [2, p. 478] states that, up to conjugacy in $GL(V_{\mathbb{L}})$ (i.e. allowing extension of scalars), G must be (a) a group of orthogonal type $O(n, q, \varepsilon)$ or $O_j(n, q, \varepsilon)$; or (b) the reduction mod p of a finite, irreducible orthogonal or unitary group generated by reflections in characteristic 0 (examples: finite Coxeter groups); or (c) one of two special groups of unitary type over finite fields, namely $[EJ_3(5)]^5$ ($n = 3$, over $GF(5^2)$), or $[J_4(4)]^3$ ($n = 4$, over $GF(3^2)$); or (d) $[\widehat{A}_n]^p \simeq S_{n+2}$, when $p \mid (n+2)$ (a slightly unfamiliar modular representation of a symmetric group).

Conside how impressive this is: if G is an infinite, irreducible linear Coxeter group, then the reduced finite group G^p can take no 'middle ground': it must either (rarely) be some finite Coxeter group, or (usually) must jump in size to some orthogonal group $O(n, q, \varepsilon)$ or $O_j(n, q, \varepsilon)$ of all (or nearly all) isometries for some form $x \cdot y$ over a finite field. This is remarkable, given the relatively small number of generators.

Egon and I [1, Thm. 3.1] ruled out the unitary cases and other odds and ends with

Theorem 0.1. *Suppose $G \subseteq GL(V)$ is a finite irreducible group generated by reflections $r_0, \ldots, r_{n-1}$, where V has dimension $n \geqslant 3$ over the finite field $\mathbb{K} = GL(q)$, of characteristic $p > 2$. Also suppose that G leaves invariant some non-zero bilinear form. Then, up to conjugacy in $GL(V_{\mathbb{L}})$, the group G is either*

(a) an orthogonal group $O(n, q, \varepsilon)$ or $O_j(n, q, \varepsilon)$, excluding the cases $O_1(3, 3, 0)$, $O_2(3, 5, 0)$, $O_2(5, 3, 0)$ (assuming for these three that disc$(V) \sim 1$), and also excluding the case $O_j(4, 3, -1)$ (all for various minor offences); or

(b) the (faithful) reduction mod p of one of the finite linear Coxeter groups generated by reflections in characteristic 0, namely the groups of type A_n, B_n, D_n, E_6 ($p \neq 3$), E_7, E_8, F_4, H_3 or H_4.

Comments. Basically we generally get after reduction either an obvious Coxeter group or a full or nearly full orthogonal group. We need to look more at the latter.

References

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