Idea: admit periods 5 and 10 to the crystallographic options $\{2, 3, 4, 6, \infty\}$ by letting the golden ratio $\tau := (1 + \sqrt{5})/2$ become an integer. (τ is the positive root of $\tau^2 = \tau + 1$.)

Recall for crystallographic G that $4\cos^2(\pi/p_{ij}) = m_{ij}m_{ji}$, a (rational) integer. Compare this with $\cos(\pi/5) = \tau/2$, so that $4\cos^2(\pi/5) = \tau^2 = \tau + 1$ and $4\cos^2(\pi/10) = \tau + 2$. If we allow such node labels, then the rotational periods 5 and 10 for $r_i r_j$ are induced by the diagrams

$$\overset{s}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\tau^2 s}{\bullet} \quad \text{and} \quad \overset{s}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{(\tau+2)s}{\bullet},$$

respectively, where now we allow rescaling of nodes by any 'integer' $s \in \mathbb{D}$ or its inverse:

Some Properties of the domain $\mathbb{D} := \mathbb{Z}[\tau] = \{a + b\tau \ : \ a, b \in \mathbb{Z}\}$, which is the ring of all algebraic integers in the field $\mathbb{Q}(\sqrt{5})$.

1. The non-trivial field automorphism mapping $\sqrt{5} \mapsto -\sqrt{5}$ induces a ring automorphism $' : \mathbb{D} \to \mathbb{D}$ (*conjugation*). Thus $(a + b\tau)' = (a + b) - b\tau$. In particular, $\tau' = 1 - \tau = -\tau^{-1}$.
2. $z = a + b\tau$ has (multiplicative) *norm* $N(z) := zz' = a^2 + ab - b^2$. And $\mathbb{D}$ is a Euclidean domain, with a division algorithm based on $|N(z)|$.
3. The set of *units* in $\mathbb{D}$ is $\{\pm\tau^n \ : \ n \in \mathbb{Z}\} = \{u \in \mathbb{D} \ : \ N(u) = \pm 1\}$. If f_n is the nth Fibonacci number, taking $f_{-1} = 1$ and $f_0 = 0$, then for all $n \in \mathbb{Z}$
$$\tau^n = f_{n-1} + f_n \tau \ .$$
4. Recall that integers $z, w \in \mathbb{D}$ are *associates* if $z = uw$ for some unit u. Up to associates, the primes $\pi \in \mathbb{D}$ can be classified as follows:
   (a) the prime $\pi = \sqrt{5} = 2\tau - 1$, which is self-conjugate (up to associates: $\pi' = -\pi$);
   (b) rational primes $\pi = p \equiv \pm 2 \bmod 5$, also self-conjugate;
   (c) primes $\pi = a + b\tau$, for which $|N(\pi)|$ equals a rational prime $q \equiv \pm 1 \bmod 5$. In this case, the conjugate prime $\pi' = (a + b) - b\tau$ is not an associate of π.

New dihedral groups. You can check directly that the pentagonal group $H_2 = I_2(5)$ really does come from the diagram $\overset{1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\tau^2}{\bullet}$ and that the decagonal group $I_2(10)$ comes from $\overset{1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{(\tau+2)}{\bullet}$. Furthermore, any 'quasicrystallographic' Coxeter group G with such a node-labelled diagram can be represented as a subgroup of $GL_n(\mathbb{D})$, through the action of G on the $\mathbb{D}$-module $\oplus_j \mathbb{D} b_j$.

Example. $G = [3,5,3]$ is a subgroup of $O(V)$, where the real 4-space V is equipped with a Lorentzian form $x \cdot y$ of signature $(+++-)$. Thus G also acts naturally on $\mathbb{H}^3$ and yields, for example, the regular icosahedral tessellation $\{3,5,3\}$ of hyperbolic space. Since τ^2 is a unit, there is essentially only one choice of diagram, namely

$$\Delta(G) = \overset{1}{\bullet} \text{------} \overset{1}{\bullet} \text{------} \overset{\tau^2}{\bullet} \text{------} \overset{\tau^2}{\bullet}\ .$$

The discriminant is

$$\mathrm{disc}(V) = -\frac{1}{16}(2+5\tau) \sim -(2+5\tau)\ ,$$

where $\delta := -(2+5\tau)$ has norm -11. Thus δ is prime in $\mathbb{D}$.

Now consider any prime $\pi \in \mathbb{D}$. We can show that $G^\pi = \langle r_0, r_1, r_2, r_3 \rangle^\pi$ is a string C-group, with regular 4-polytope $\mathcal{P}^\pi$.

The subgroup $G_3^\pi = \langle r_0, r_1, r_2 \rangle^\pi$ is obviously some quotient of the spherical group $[3,5] \simeq H_3$. After reduction modulo any prime π, even for associates of 2, the reflections r_j still have period 2. And consider

$$z := (r_0 r_1 r_2)^5 = \begin{bmatrix} -1 & 0 & 0 & \tau^4 \\ 0 & -1 & 0 & 2\tau^4 \\ 0 & 0 & -1 & 3\tau^2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\ . \tag{1}$$

Since τ^4 is a unit, $r_0 r_1 r_2$ still has period 10 in G^π. Thus $\langle r_0, r_1, r_2 \rangle^\pi \simeq [3,5]$ and dually $\langle r_1, r_2, r_3 \rangle^\pi \simeq [5,3]$.

Netx we emply [4, Th. 4.2], which essentially says that so long as (say) the facet group G_3^π is a Coxeter group of spherical type and the vertex-figure group G_3^π is any string C-group, then G^π itself must be a string C-group. The key step in proving this is to show that the orbit of $\mu_0 := [1,0,0,0]$ under the right action of the matrix group $\langle r_0, r_1, r_2 \rangle$ has the same size modulo π as in characteristic 0, namely 12. (These row vectors correspond naturally to the vertices of the base icosahedron in $\{3,5,3\}$.) This is a routine check, so we get

Proposition 0.1. *Let $G = [3,5,3]$. For any prime $\pi \in \mathbb{D}$, the group $G^\pi = \langle r_0, r_1, r_2, r_3 \rangle^\pi$ is a finite string C-group. The corresponding finite regular polytope $\mathcal{P}^\pi$ is self-dual and has icosahedral facets $\{3,5\}$ and dodecahedral vertex figures $\{5,3\}$.*

A more detailed description of G^π depends on the nature of the prime π. In all cases the underlying finite field $\mathbb{K} := \mathbb{D}/(\pi)$ has order $|N(\pi)|$, and G^π acts as an orthogonal group on a 4-dimensional vector space V over $\mathbb{K}$.

Case 1: $\pi = 2$. Here an easy calculation using GAP confirms that G^π is the orthogonal group $O(4, 2^2, -1)$ with Witt index 1 over $\mathbb{K} = GF(2^2)$. Since $|G^2| = 8160$, the polytope $\mathcal{P}^2$ has 68 vertices and 68 icosahedral facets.

Case 2: $\pi = \sqrt{5} = 2\tau - 1$. Here $|\pi\pi'| = 5 \equiv 0 \bmod \pi$, so that the discriminant $\delta = -(2+5\tau) \equiv 3 \bmod \pi$, which is non-square in $\mathbb{K} = GF(5)$. Thus $\varepsilon = -1$ and $G^\pi = O_1(4, 5, -1)$ has order 15600.

Case 3: π is an associate of an odd rational prime $p \equiv \pm 2 \bmod 5$. Now $G^p = O_1(4, p^2, \varepsilon)$, where

$$\varepsilon := \begin{cases} +1, & \text{if } p \equiv 3, 12, 23, 27, 37, 38, 42, 47, 48, 53 \bmod 55; \\ -1, & \text{if } p \equiv 2, 7, 8, 13, 17, 18, 28, 32, 43, 52 \bmod 55. \end{cases}$$

Case 4: $\pi = a + b\tau$, with norm $N(\pi) = a^2 + ab - b^2 = q$, where the rational prime $q \equiv \pm 1 \bmod 5$. (But assume π is not an associate of $\delta = -(2 + 5\tau)$.) Now $G^\pi = O_1(4, q, \varepsilon)$, where $\varepsilon = (\,(5ab - 2b^2)\,|\,q\,)$ can be computed using the rational Legendre symbol.

Case 5 – the most interesting!: $\pi = \delta = -(2 + 5\tau)$, the only case in which the orthogonal space V is singular. Now $\mathbb{K} = GF(11)$ and $\tau = -2/5 = 4$. We find that $\mathrm{rad}(V)$ is spanned by $c = 7b_0 + 3b_1 + 2b_2 + b_3$, and that $V = \mathrm{rad}(V) \perp V_3$, where V_3 is the non-singular subspace spanned by b_0, b_1, b_2. Then $O(V) \simeq \check{V}_3 \rtimes (\mathbb{K}^* \times O(V_3))$, and the subgroup

$$G^\delta = \widehat{O}_1(V) \simeq \check{V}_3 \rtimes O_1(V_3)$$

has order $11^3 \cdot 11 \cdot (11^2 - 1)$. The isometry $z = (r_0 r_1 r_2)^5$ now acts as the *central inversion* in the group $O_1(V_3)$ for the icosahedral facet. And G^δ has a normal subgroup A isomorphic to $\check{V}_3 \rtimes \langle z \rangle$ with order $2 \cdot 11^3$. Using the fact that $O_1(3, 11, 0) \simeq PSL_2(11) \rtimes C_2$, we conclude that

$$\overline{G} := G^\delta / A \simeq PSL_2(11) \ ,$$

of order 660. Remarkably, $\overline{G}$ is also a string C-group. The resulting polytope is the 11-cell independently discovered by Coxeter in [3] and Grünbaum in [1]. Indeed, both $r_0 r_1 r_2$ and $r_1 r_2 r_3$ have period 5 in the quotient and

$$\mathcal{P}(\overline{G}) = \{\, \{3, 5\}_5 \, , \, \{5, 3\}_5 \,\}$$

is the universal 4-polytope with hemi-icosahedral facets and hemi-dodecahedral vertex-figures.

A similar analysis is possible for the group $H = [5, 3, 5]$ with diagram

$$\Delta(H) = \overset{1}{\bullet} \text{———} \overset{\tau^2}{\bullet} \text{———} \overset{\tau^2}{\bullet} \text{———} \overset{1}{\bullet}$$

and corresponding discriminant $\frac{-1}{16}(3 + 7\tau) \sim -(3 + 7\tau) =: \lambda$. Since $N(\lambda) = -19$, we see that λ is also prime. The group H^λ for the singular space V again has an interesting quotient, giving

$$\overline{H} \simeq PSL_2(19)$$

as the automorphism group for the universal regular polytope

$$\mathcal{P}(\overline{H}) = \{\, \{5, 3\}_5 \, , \, \{3, 5\}_5 \,\} \ ,$$

with hemi-dodecahedral facets and hemi-icosahedral vertex-figures. This is the 57-cell described by Coxeter in [2].

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