Amalgamation of Groups

We basically follow Bourbaki, Algebra - I, §7.3 [1].

1. Let $\{M_i : i \in I\}$ be a family of groups. For another group A we suppose that we have injective homomorphisms

$$h_i : A \to M_i \ .$$

Note that we may assume that A and the M_i's are mutually disjoint.

Let P_i be a right transversal of $(A)h_i$ in M_i, with $P_i \cap (A)h_i = \{e_i\}$, the identity in M_i. For each $z \in M_i$ there are unique $a \in A$ and $p \in P_i$ with

$$z = (a)h_i\, p \ .$$

2. **Presentations**.

Suppose that $M = \langle X_i \,|\, R_i \rangle$. Once more we may assume that the X_i's are mutually disjoint. Thus

$$M_i \simeq \mathrm{Fr}(X_i)/\mathrm{N}(R_i) \ .$$

Let $\pi_i : \mathrm{Fr}(X_i) \to M_i$ be the natural map, so that $\ker(\pi_i) = \mathrm{N}(R_i)$ is the normal subgroup generated by the relator set R_i in $\mathrm{Fr}(X_i)$.

Let $X = \cup_i X_i$. For each i there is a natural injection $\mathrm{Fr}(X_i) \hookrightarrow \mathrm{Fr}(X)$. In fact, if we construct free groups in the usual way from sets of reduced words, we can even assume that each $\mathrm{Fr}(X_i)$ is a *subgroup* of $\mathrm{Fr}(X)$ (usually not of finite index).

Now specify a set A_{gen} of generators for A. For each $a \in A_{gen}$ and $i \in I$, choose a word $w(a,i) \in \mathrm{Fr}(X_i) \subseteq \mathrm{Fr}(X)$ such that

$$w(a,i)\pi_i = (a)h_i \in M_i \ .$$

Next take

$$R = \bigcup_i R_i \ \cup \ \{w(a,i)\, w(a,j)^{-1} \ : \ a \in A_{gen}, i, j \in I\} \ .$$

Definition 0.1. *The* amalgamated product *of the groups M_i along the subgroup A is*

$$M := \mathrm{Fr}(X)/\mathrm{N}(R) \ .$$

Let $\pi : \mathrm{Fr}(X) \to M$ be the natural map.

Remarks. The group M may be denoted

$$*_A M_i \ ,$$

although this suppresses the crucial nature of the isomorphisms h_i.

The special relations in R effectively identify all copies of A in the M_i.

3.

Theorem 0.2. [**Pushouts**] *M is universal for families of maps $t_i : M_i \to G$ which agree on the subgroups $(A)h_i$:*

$$
\begin{tikzcd}
 & & M \arrow[dd, dashed, "\exists!\, f"] \\
 & M_i \arrow[ur, "\varphi_i"] \arrow[dr, "t_i"] & \\
A \arrow[ur, "h_i"] \arrow[dr, "h_j"'] & \vdots & G \\
 & M_j \arrow[uuur, "\varphi_j"] \arrow[ur, "t_j"'] &
\end{tikzcd}
$$

If $h_i t_i = h_j t_j : A \to G$ for all $i, j \in I$, then there exists a unique homomorphism $f : M \to G$ such that $\varphi_i f = t_i$ for all $i \in I$.

Proof.

(a) We may define

$$
\begin{array}{rcl}
\varphi_i : M_i & \to & M \\
x & \mapsto & x, \quad x \in X_i
\end{array}
$$

since each $R_i \subseteq R$. We don't yet know that φ_i is injective, because of the special relations in R.

(b) Those special relations do guarantee that in M we have

$$(a)h_i\,\varphi_i = (a)h_j\,\varphi_j$$

for all $i, j \in I$ and $a \in A$. Thus we have a single map

$$h = h_i \varphi_i : A \to M$$

for all $i \in I$.

(c) Now suppose $h_i t_i = h_j t_j : A \to G$ for all $i, j \in I$ and some general group G. Clearly M is generated by its subgroups $(M_i)\varphi_i$, so we are forced to define $f : M \to G$ so that $(x)\pi_i \varphi_i f = (x)\pi_i t_i$, for all $x \in X_i$. But since t_i is a homomorphism, all relations from R_i are satisfied when transferred to G. The remaining relations in R have the form $w(a, i)w(a, j)^{-1} = 1$. These are satisfied in G since $(a)h_i t_i = (a)h_j t_j$. Thus f is uniquely well-defined. $\square$

Remark. We have that the universal object M exists. By standard arguments, M is unique to isomorphism, given the injective maps $h_i : A \to M_i$.

For more detailed structure, we must exploit explicit representations of M.

(d) Motivated by 'intuitive' calculations in M, we now consider finite sequences

$$\sigma = (a; i_1, \ldots, i_n; p_1, \ldots, p_n)$$

where $a \in A$, $i_\alpha \in I$ and $p_\alpha \in P_{i_\alpha}$, for $1 \leqslant \alpha \leqslant n$. We say

- σ has length $l(\sigma) = n$. For $n = 0$, $\sigma = (a)$.
- σ is a *decomposition* for $x \in M$ if

$$x = (a)h \cdot \prod_{\alpha=1}^{n} (p_\alpha)\varphi_{i_\alpha}$$

- σ is *reduced* if $i_\alpha \neq i_{\alpha+1}$, for $1 \leqslant \alpha < n$ and $p_\alpha \neq e_{i_\alpha}$ for $1 \leqslant \alpha \leqslant n$.
  In particular, if e is the identity in A, then (e) is a reduced decomposition for the identity $1 \in M$. More generally, (a) is a reduced decomposition for $(a)h \in M$.

(e) Let Σ be the set of all reduced decompositions σ. Define

$$\begin{array}{rcl} \Phi : \Sigma & \to & M \\ \sigma = (a; i_1, \ldots, i_n; p_1, \ldots, p_n) & \mapsto & (a)h \cdot \prod_{\alpha=1}^{n} (p_\alpha)\varphi_{i_\alpha} \end{array}$$

(We want to show that Φ is a bijection.) For fixed i let

$$\Sigma_i := \{\sigma \in \Sigma \,:\, \sigma = (e; i_1, \ldots, i_n; p_1, \ldots, p_n), \text{where } i_1 \neq i, \text{if } n \geqslant 1\} .$$

Now we may define

$$\Psi_i : M_i \times \Sigma_i \to \Sigma$$

by

$$(z, \sigma) \mapsto \begin{array}{ll} (a; i_1, \ldots, i_n; p_1, \ldots, p_n), & \text{if } p = e_i; \\ (a; i, i_1, \ldots, i_n; p, p_1, \ldots, p_n), & \text{if } p \neq e_i, \end{array}$$

where $\sigma = (e; i_1, \ldots, i_n; p_1, \ldots, p_n)$ and $z = (a)h_i\, p$, with $a \in A$ and $p \in P_i$. It is easy to check that Ψ_i is well-defined and onto.

On the other hand, suppose $(z, \sigma)\Psi_i = (z', \sigma')\Psi_i$, for $\sigma' = (e; i'_1, \ldots, i'_m; p'_1, \ldots, p'_m)$ and $z' = (a')h_i\, p'$, with $a' \in A$ and $p' \in P_i$. Then $a = a'$. If $p = e_i$, then $i_1 \neq i$, so $p' = e_i$, $m = n$ and indeed $\sigma = \sigma'$. If $p \neq e_i$, then similarly we must have $\sigma = \sigma'$.

(f) Thus each Ψ_i is bijective. This lets use define for each $i \in I$ and $x \in M_i$ a mapping

$$\begin{aligned} f_{i,x} : \Sigma &\to \Sigma \\ (z,\sigma)\Psi_i &\mapsto (x^{-1}z,\sigma)\Psi_i \end{aligned}$$

Clearly $f_{i,x}$ is bijective.
For each $a \in A$ we can also define

$$\begin{aligned} f_a : \Sigma &\to \Sigma \\ (a'; i_1, \ldots, i_n; p_1, \ldots, p_n) &\mapsto (a^{-1}\, a'; i_1, \ldots, i_n; p_1, \ldots, p_n) \end{aligned}$$

Clearly f_a is also bijective.
It is easy to check that

- $f_{i,e_i} = 1_\Sigma$
- $f_{i,xy} = f_{i,x} f_{i,y}$
- for any $a \in A$ and $i \in I$, we have $f_a = f_{i,(a)h_i}$.

(g) Now let $G = \mathrm{Sym}(\Sigma)$. We have homomorphisms

$$\begin{aligned} t_i : M_i &\to G \\ x &\mapsto f_{i,x} \end{aligned}$$

such that for all $a \in A$ and $i \in I$,

$$(a)h_i t_i = f_{i,(a)h_i} = f_a \ ,$$

independent of i. Thus there is a unique map $f : M \to G$ with $\varphi_i f = t_i$ for all i. Now let

$$\sigma = (a; i_1, \ldots, i_n; p_1, \ldots, p_n)$$

- In M_i we have $e_i = e_i\, e_i = (e)h_i\, e_i$. Thus if $\sigma = (e; i_1, \ldots, i_n; p_1, \ldots, p_n) \in \Sigma_i$, we must have $(e_i, \sigma)\Psi_i = (e; i_1, \ldots, i_n; p_1, \ldots, p_n) = \sigma$. In particular, $(e_i, (e))\Psi_i = (e)$.
- Suppose $p_1 \in P_{i_1} - \{e_{i_1}\}$. Then
$$(e) f_{i_1, p_1^{-1}} = (p_1, (e))\Psi_{i_1} = (e, i_1; p_1) \ .$$
- By induction we get
$$\begin{aligned} (e) f_{i_n, p_n^{-1}}\, f_{i_{n-1}, p_{n-1}^{-1}} \ldots f_{i_1, p_1^{-1}} &= (e; i_2, \ldots, i_n; p_2, \ldots, p_n) f_{i_1, p_1^{-1}} \\ &= (p_1, (e; i_2, \ldots, i_n; p_2, \ldots, p_n))\Psi_{i_1} \\ &= (e; i_1, i_2, \ldots, i_n; p_1, p_2, \ldots, p_n) \ . \end{aligned}$$

  Thus
$$(e) f_{i_n, p_n^{-1}}\, f_{i_{n-1}, p_{n-1}^{-1}} \ldots f_{i_1, p_1^{-1}} f_{a^{-1}} = (a; i_1, \ldots, i_n; p_1, \ldots, p_n) \ .$$

(h) Note that $f_{i,p} = (p)t_i = (p)\varphi_i f$ and

$$((a^{-1})h)f = ((a^{-1})h_i\varphi_i)f = (a^{-1})h_i t_i = f_{a^{-1}}$$

for all i. Thus for any $\sigma = (a; i_1, \ldots, i_n; p_1, \ldots, p_n)$ we have

$$(e)[(p_n^{-1})\varphi_{i_n} \cdots (p_1^{-1})\varphi_{i_1}(a^{-1})h]f = \sigma \ ,$$

or

$$(e)[(\sigma\Phi)^{-1}]f = \sigma \ .$$

Thus $\Phi : \Sigma \to M$ is injective: **reduced decompositions are unique**.

4. **Existence**

Let D be the set of all elements of M which admit decompositions. Thus $1 \in D$. Now M is generated by

$$(A)h \cup \bigcup_i (P_i)\varphi \ .$$

Lemma 0.3. Shifting. *Say $i_\alpha \in I$, $p_\alpha \in P_{i_\alpha}$, for $1 \leqslant \alpha \leqslant n$. For each $a \in A$ there exist $a' \in A$ and $p'_\alpha \in P_{i_\alpha}$, $1 \leqslant \alpha \leqslant n$, such that*

$$(p_1)\varphi_{i_1} \ldots (p_n)\varphi_{i_n}(a)h = (a')h(p'_1)\varphi_{i_1} \ldots (p'_n)\varphi_{i_n} \ .$$

Moreover, $p_\alpha \neq e_{i_\alpha} \Rightarrow p'_\alpha \neq e_{i_\alpha}$.

Proof. (Induction) Since $(a)h = (a)h_{i_n}\varphi_{i_n}$, there exist $a_n \in A$ and $p'_n \in P_{i_n}$ such that $p_n(a)h_{i_n} = (a_n)h_{i_n}p'_n$. Thus $(p_n)\varphi_{i_n}(a)h = (a_n)h(p'_n)\varphi_{i_n}$. □

It follows that $D \cdot (A)h \subseteq D$. Similarly, $D \cdot (P_i)\varphi_i \subseteq D$. The case $i_n \neq i$ is trivial; the case $i_n = i$ follows as in the Lemma.

5. **Conclusion** Each $x \in M$ has a *unique reduced decomposition* σ, so that

$$x = (a)h \prod_{\alpha=1}^{n} (p_\alpha)\varphi_{i_\alpha} \ ,$$

with $a \in A$ and $p_\alpha \in P_{i_\alpha} - \{e_{i_\alpha}\}$. Any other decomposition σ' for x must have $l(\sigma') > l(\sigma)$, since any non-reduced decomposition can be progressively shortened using Lemma 0.3 to yield a reduced decomposition.

6. **Consequences**

(a) $h = h_i\varphi_i$ is injective.
If $(a)h = (a')h = x$ in M, then (a) and (a') are both reduced decompositions for x, so $a = a'$ by uniqueness.

(b) Each $\varphi_i : M_i \to M$ is injective.

Since h is injective, we have that φ_i must be injective on $(A)h_i \subseteq M_i$. Now suppose $(x)\varphi_i = (x')\varphi_i$ for $x = (a)\, h_i p, \quad x' = (a')\, h_i p'$, where $p, p' \in P_i$ and $a, a' \in A$. Thus $z = (a)h \cdot (p)\varphi_i = (a')h \cdot (p')\varphi_i \in M$ has two decompositions: $(a; i; p)$ [or (a), when $p = e_i$] and $(a'; i; p')$ [or (a'), when $p' = e_i$]. By uniqueness, in any of these cases we have $a = a'$ and $p = p'$, so $x = x'$.

(c) For $i \neq j$, $(M_i)\varphi_i \cap (M_j)\varphi_j = (A)h$.

Clearly $(A)h = (A)h_i\varphi_i \subseteq (M_i)\varphi_i$. So suppose $z \in (M_i)\varphi_i \cap (M_j)\varphi_j$. Then z has reduced decompositions $(a; i; p)$ [or (a), when $p = e_i$] and $(a'; j; p')$ [or (a'), when $p' = e_j$]. By uniqueness this means that $p = e_i$, $p' = e_j$ and $a = a'$. Thus $z = (a)h$.

(d) **Conventions**. After suitable identifications we can assume that $M_i \subseteq M$ for all $i \in I$, $A \subseteq M$, M is generated by the M_i's and $M_i \cap M_j = A$, for all $i \neq j$.

(e) Suppose $x \in M$ has reduced decomposition

$$\sigma = (a; i_1, \ldots, i_n; p_1, \ldots, p_n)\ .$$

Considering the above identifications we may write

$$x = ap_1 \cdots p_n$$

where $a \in A$, $p_\alpha \in P_{i_\alpha} - \{1\}$, with all $i_\alpha \neq i_{\alpha+1}$.

If $i_1 \neq i_n$, then

$$\begin{aligned} x^2 &= (ap_1 \cdots p_n)(ap_1 \cdots p_n) \\ &= (aa')(p_1' \cdots p_n')(p_1 \cdots p_n) \end{aligned}$$

is also reduced. By induction we conclude that x must have infinite period.

On the other hand, if $i_1 = i_n$, then $p_n x p_n^{-1} = (p_n a p_1) \cdot p_2 \cdots p_{n-1}$, where $p_n a p_1 = a' p_1'$ for $p_1' \in P_{i_1}$, $a' \in A$.

Thus $p_n x p_n^{-1}$ has length $n-1$ or $n-2$ depending on whether $p' \neq e_{i_1}$ or $p' = e_{i_1}$.

Thus: every element of finite order in M is conjugate to an element of finite order in some particular M_j.

7. **Well-positioned Subgroups**. Continuing with the same conventions, suppose that we have subgroups $N_i < M_i$ for each i and a fixed subgroup $B < A$ such that

$$N_i \cap A = B, \text{ for each } i \ .$$

Lemma 0.4. *For each i a transversal $\tilde{P}_i$ to B in N_i can be extended to a transversal P_i to A in M_i.*

Proof. For $x, y \in \tilde{P}_i$ suppose $Ax = Ay$. Since $x, y \in N_i$, we have $xy^{-1} \in A \cap N_i = B$, so $x = y$. □

Theorem 0.5. (Bourbaki, p. 167) *The subgroup of M generated by the N_i's is isomorphic to*

$$N := *_B N_i \ .$$

Proof. Choose transversals as in the Lemma. We have inclusions $s_i : N_i \hookrightarrow M$ which obviously agree on B. Let $\psi_i : N_i \to N$ be the canonical maps (which we have proved must be injections). By the universal property of N there exists a unique map $g : N \to M$ such that $\psi_i g = s_i$ for all i.

Now let $z \in \ker(g)$. As an element of N, z thus has a canonical form $z = \tilde{b}\,(\tilde{t}_1)\psi_{i_1} \cdots (\tilde{t}_n)\psi_{i_n}$, where $\tilde{b} = b\psi_i$ for all i and the $t_j \in \tilde{P}_{i_j} \subseteq P_{i_j}$, say. Applying the inclusions s_{i_j} we get

$$1 = (z)g = b\tilde{t}_1 \cdots \tilde{t}_n \ .$$

Since (by our choice of transversals) this is a canonical form in M, we must have $n = 0$ and $b = 1$. Thus $z = 1$ and g is injective. □

References

[1] N. Bourbaki, *Elements of Mathematics, Algebra I*, Hermann, Paris, (Addison-Wesley, Reading, Mass.), 1974.