Recap: The general abstract Coxeter group with n generators is

$$\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \, : \, (\rho_i \rho_j)^{p_{ij}} = 1, \quad 0 \leqslant i, j \leqslant n-1 \rangle,$$

where $p_{ii} = 1$ and $2 \leqslant p_{ij} = p_{ji} \leqslant \infty$ for all $i \neq j$. Recall V has real basis $a_0, \ldots, a_{n-1}$ and symmetric bilinear form specified by $a_i \cdot a_j := -2 \cos \pi / p_{ij}$. Thus $r_j(x) = x - (x \cdot a_j) a_j$, and $G = \langle r_0, \ldots, r_{n-1} \rangle$ is a subgroup of $O(V)$.

Theorem (Bourbaki, i.e. Tits?) [2, Ch. 5.3-5.4] The mapping $\rho_j \mapsto r_j$ induces a faithful representation $R : \Gamma \to G$. (By this 'standard representation' we often identify Γ and G.)

Outline of Proof.

1. **Step 1**: prove that the ρ_j have period 2, using the multiplicative group $\{\pm 1\}$. From the Substitution Theorem (von Dyck) mapping all $\rho_j \mapsto -1$ induces an epimorphism

$$\mathrm{sgn} : \Gamma \to \{\pm 1\}.$$

2. **Step 2**: examine the length function $l(\gamma)$ (w.r.t generators $\rho_0, \ldots, \rho_{n-1}$).

   Definition: $\gamma \in \Gamma$ has length $l(\gamma) = k$ if $\gamma = \rho_{i_1} \cdots \rho_{i_k}$ where k is minimal (so this word in the generators is reduced).

   Check: $l(\gamma) = 0$ if–f $\gamma = 1$; $l(\gamma) = 1$ if–f $\gamma = \rho_j$ for some j; $l(\gamma) = l(\gamma^{-1})$;
   $l(\gamma\alpha) \leqslant l(\gamma) + l(\alpha)$ (triangle inequality).

   Exercise: $\mathrm{sgn}(\gamma) = (-1)^{l(\gamma)}$ for all $\gamma \in \Gamma$.

   It follows that $l(\gamma\rho_j) = l(\gamma) \pm 1$, for $\gamma \in \Gamma$ and any ρ_j.

   Remark: determining when $+$, when $-$ is crucial; relates to how mirror for ρ_j positioned relative to the (geometrical) flag associated to γ.

3. **Step 3**: Introduce $R : \Gamma \to G$, which induces an action of Γ on V: $\gamma(x) = R(\gamma)(x)$, for $\gamma \in \Gamma, x \in V$.

   The fact that R is well-defined again uses the Substitution Theorem. Consider the relation $(\rho_i\rho_j)^{p_{ij}}$ for $i \neq j$. The bilinear form induced in the plane spanned by a_i, a_j in V is precisely that which we encountered in Lecture 2, when we observed that the dihedral reflection group $I_2(q)$ acts faithfully in a familiar way on $\mathbb{R}^2$. Now check that the period of $r_i r_j$ is determined by its action on the plane spanned by a_i, a_j.

4. **Step 4**: study the root system

$$\Phi = \cup_{j=0}^{n-1} \Gamma a_j$$

   (a union of orbits). Think of the set of all mirror normals; note all have same length $\sqrt{2}$.

For $b \in \Phi$ write $b = \sum_{j=0}^{n-1} t_j a_j$ and define

$b > 0$ if all $t_j \geqslant 0$; eg. $a_j > 0$, so $a_j \in \Phi^+$.

$b < 0$ if all $t_j \leqslant 0$; eg. $\rho_j(a_j) = -a_j < 0$, so $-a_j \in \Phi^-$.

** Not clear every root $b \in \Phi$ is in fact one of + or −!!

5. **Step 5**: 1st Delicate Proposition [2, Thm. 5.4]. Let $\gamma \in \Gamma$. Then

$$\begin{array}{lll} l(\gamma\rho_j) > l(\gamma) & \text{implies} & \gamma(a_j) > 0 \\ l(\gamma\rho_j) < l(\gamma) & \text{implies} & \gamma(a_j) < 0 \end{array} \tag{1}$$

Note how both situations can happen. We are relating an abstract length function to concrete geometry.

6. **Step 6**: Consequences of 1st Delicate Proposition.

   (a) Every root is + or −: $\Phi = \Phi^+ \sqcup \Phi^-$

   (b) The representation R is faithful. Yes!

   (c) Γ satisfies the *intersection condition*. For any $I \subseteq \{0, \ldots, n-1\}$, let

   $$\Gamma_I := \langle \rho_k \ : \ k \in I \rangle$$

   (a *parabolic* subgroup of Γ). Then for all $I, J \subseteq \{0, \ldots, n-1\}$, we have

   $$\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J} \ . \tag{2}$$

   This is true for any Coxeter group, regardless of the topology of its diagram.

   (d) Γ_I is a Coxeter group. Its diagram is the subdiagram on the node set I. Its intrinsic length function $l_I = l$ on Γ_I (a kind of convexity condition in the Coxeter complex). In other words, one cannot reduce the word length by moving from I to $\{0, \ldots, n-1\}$.

7. **Step 7**: Proof of 1st Delicate Proposition

The first possibility, i.e.

$$l(\gamma\rho_j) > l(\gamma) \text{ implies } \gamma(a_j) > 0,$$

actually forces the second. We use induction on $k = l(\gamma)$. Since $\gamma = 1$ if $k = 0$, and $a_j > 0$, the case $k = 0$ is settled. So suppose $k \geqslant 1$ and $\gamma = \rho_{i_1} \cdots \rho_{i_k}$ is a reduced word. Then $l(\gamma\rho_{i_k}) = k - 1 < k = l(\gamma)$, so $j \neq i_k$. Now look very carefully at the dihedral group $\langle r_j, r_{i_k} \rangle = \Gamma_{j,i_k}$ acting on the plane spanned by a_j and a_{i_k}. In [2, 5.4] we see how to choose a special coset representative in $\gamma\langle r_j, r_{i_k} \rangle$. We look carefully at the length function $l_{\{j,i_k\}}$ in this dihedral group. The argument in the finite case $p_{j,i_k} < \infty$ is finicky, but really just amounts to seeking a shortest path along edges of the regular polygon.

8. **Step 8**: Read [2] or [1]. There is much, much more to learn here.

References

[1] N. Bourbaki, *Groupes et Algébres de Lie, Chapitres IV-VI*, Hermann, Paris, 1968.

[2] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, UK, 1990.