The Tomotope

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Abstract

Every abstract 3-polytope $\mathcal{M}$, in particular, every polyhedral map, has a unique minimal regular cover, and the automorphism group of this cover is isomorphic to the monodromy group of $\mathcal{M}$. Here we demonstrate that the situation for polytopes of higher rank must be very different: the tomotope $\mathcal{T}$ is a small, highly involved, abstract uniform 4-polytope. It has infinitely many distinct minimal regular covers.

1 Introduction

The monodromy group is a most useful tool when one tries to understand combinatorial properties of maps, such as coverings, automorphism groups, flag orbits, stellations, and so forth; see [10, §3] and [8], for example.

Changing our structural viewpoint a little, we may consider polyhedral maps, at least, to be abstract 3-polytopes. This paper arose as a by-product of a wider effort [17] to understand basic combinatorial constructions for abstract polytopes in all ranks. In this setting, the monodromy group already appeared in [5] (as the image of the 'flag action').

One of our concerns is to understand how an n-polytope $\mathcal{P}$, with no particular attributes of symmetry, can be covered by a regular n-polytope $\mathcal{R}$. When $\mathcal{P}$ has rank $n = 3$, there is a minimal such regular cover, intimately related to the monodromy group of $\mathcal{P}$. (This fact seems to be well-known, but we must refer to [17] for an easy proof.) On the other hand, our attempts to extend the result to higher ranks are thwarted by that best of all reasons, a counterexample. Thus we come to the *tomotope*, a most peculiar uniform 4-polytope $\mathcal{T}$, constructed in Section 4 below. We shall prove in Theorem 5.9 that $\mathcal{T}$ has infinitely many distinct and 'incommensurable' minimal regular covers.

We have given $\mathcal{T}$ its name as a small, and rather late, birthday present for Tomaž Pisanski, in particular gratitude for his insights and enthusiasm. In fact, Tomo himself suggested that Reye's configuration might be hidden in $\mathcal{T}$ (see Section 6).

2 Abstract Polytopes and their Monodromy Groups

An abstract n-polytope $\mathcal{P}$ has some of the key combinatorial properties of the face lattice of a convex n-polytope; in general, however, $\mathcal{P}$ need not be a lattice, need not be finite, need not have any familiar geometric realization. An important example for us will be the familiar tiling $\mathcal{U}$ of ordinary space by regular octahedra and tetrahedra, the beginning of which is displayed in Figure 1. An indication that $\mathcal{U}$ is indeed an abstract uniform 4-polytope is that each 'flag' has four entries, namely a mutually incident vertex, edge, triangle and cell (octahedron or tetrahdron).

Next let us review some general definitions and results, referring to [13] for details. An *abstract n-polytope* $\mathcal{P}$ is a partially ordered set with properties **A**, **B** and **C** below. A *pre-polytope* need only satisfy the first two requirements:

A: $\mathcal{P}$ has a strictly monotone rank function with range $\{-1, 0, \ldots, n\}$.

An element $F \in \mathcal{P}$ with $\operatorname{rank}(F) = j$ is called a *j-face*; typically F_j will indicate a j-face. Moreover, $\mathcal{P}$ has a unique least face F_{-1} and unique greatest face F_n. Each maximal chain or *flag* in $\mathcal{P}$ therefore contains $n+2$ faces. We let $\mathcal{F}(\mathcal{P})$ be the set of all flags in $\mathcal{P}$.

Naturally, faces of ranks 0, 1 and $n-1$ are called vertices, edges and facets, respectively.

B: Whenever $F < G$ with $\operatorname{rank}(F) = j-1$ and $\operatorname{rank}(G) = j+1$, there are exactly two j-faces H with $F < H < G$.

For $0 \leqslant j \leqslant n-1$ and any flag Φ, there thus exists a unique *adjacent* flag Φ^j, differing from Φ in just the face of rank j. With this notion of adjacency, $\mathcal{F}(\mathcal{P})$ becomes the *flag graph* for $\mathcal{P}$. Whenever $F \leqslant G$ are incident faces in $\mathcal{P}$, the *section*

$$G/F := \{H \in \mathcal{P} \,|\, F \leqslant H \leqslant G\} \,.$$

C: $\mathcal{P}$ is *strongly flag–connected*, that is, the flag graph for each section is connected.

It follows that G/F is a $(k-j-1)$-polytope in its own right, whenever $F \leqslant G$ with $\operatorname{rank}(F) = j \leqslant k = \operatorname{rank}(G)$. For example, if F is a vertex, then the section F_n/F is called the *vertex-figure* over F.

The *automorphism group* $\Gamma(\mathcal{P})$ consists of all order-preserving bijections on $\mathcal{P}$. We say $\mathcal{P}$ is *regular* if $\Gamma(\mathcal{P})$ is transitive on the flag set $\mathcal{F}(\mathcal{P})$. In this case we may choose any one flag $\Phi \in \mathcal{F}(\mathcal{P})$ as *base flag*, then define ρ_j to be the (unique) automorphism mapping Φ to Φ^j, for $0 \leqslant j \leqslant n-1$. From [13, 2B] we recall that $\Gamma(\mathcal{P})$ is then a *string C-group*, meaning that it has the following properties **SC1** and **SC2**:

SC1: $\Gamma(\mathcal{P})$ is a *string group generated by involutions* (*sggi*), that is, it is generated by involutions $\rho_0, \ldots, \rho_{n-1}$ which satisfy the commutativity relations typical of a Coxeter group with string diagram, namely

$$(\rho_j \rho_k)^{p_{jk}} = 1, \text{ for } 0 \leqslant j \leqslant k \leqslant n-1, \tag{1}$$

where $p_{jj} = 1$ and $p_{jk} = 2$ whenever $|j - k| > 1$.

SC2: $\Gamma(\mathcal{P})$ satisfies the *intersection condition*

$$\langle I \rangle \cap \langle J \rangle = \langle I \cap J \rangle, \text{ for any } I, J \subseteq \{\rho_0, \ldots, \rho_{n-1}\} . \tag{2}$$

The fact that one can reconstruct a regular polytope in a canonical way from any string C-group Γ is at the heart of the theory [13, 2E]. We will see later that the monodromy group of $\mathcal{P}$ is always an sggi but might not be a string C-group.

The periods $p_j := p_{j-1,j}$ in (1) are assembled into the *Schläfli* symbol $\{p_1, \ldots, p_{n-1}\}$ for the regular polytope $\mathcal{P}$. We note that every 2-polytope or *polygon* $\{p_1\}$ is automatically regular; its automorphism group is dihedral of order $2p_1$.

There are various ways to relax symmetry and thereby broaden the class of groups $\Gamma(\mathcal{P})$. For example, we shall agree that any polytope of rank at most 2 is *uniform* (as well as regular), then inductively define $\mathcal{P}$ to be uniform if its facets are uniform and its symmetry group is transitive on vertices [2, 11]. Thus, the abstract uniform polytopes $\mathcal{P}$ form a huge, perhaps untamable, class of mostly unfamiliar objects, though certainly including all regular polytopes.

Next we describe some tools for describing how one polytope can cover another.

Definition 2.1. *[13, 2D] Let $\mathcal{R}$ and $\mathcal{P}$ be pre-polytopes, both of rank n. A rap-map is a rank and adjacency preserving homomorphism $\eta : \mathcal{R} \to \mathcal{P}$. (This means that η induces a mapping $\mathcal{F}(\mathcal{R}) \to \mathcal{F}(\mathcal{P})$ which sends any j-adjacent pair of flags in $\mathcal{R}$ to another such pair in $\mathcal{P}$.) A surjective rap-map is called a covering; we then say $\mathcal{R}$ is a cover of $\mathcal{P}$ and write $\mathcal{R} \to \mathcal{P}$. The cover $\mathcal{R}$ is minimal (for $\mathcal{P}$) if $\mathcal{R} \neq \mathcal{P}$ and $\mathcal{R} \to \mathcal{Q} \to \mathcal{P}$ implies $\mathcal{Q} = \mathcal{R}$ or $Q = \mathcal{P}$.*

Remark 2.2. *Note that if $\mathcal{P}$ is flag-connected, as is the case for all polytopes, then $\eta : \mathcal{R} \to \mathcal{P}$ must surjective, hence a covering.* □

In order to understand how $\mathcal{P}$ arises by identifications in $\mathcal{R}$, we modify Hartley's approach in [5] and instead exploit the monodromy group.

Definition 2.3. *Let $\mathcal{P}$ be a polytope of rank $n \geqslant 1$. For $0 \leqslant j \leqslant n-1$, let s_j be the bijection on $\mathcal{F}(\mathcal{P})$ which maps each flag Φ to the j-adjacent flag Φ^j. Then the monodromy group for $\mathcal{P}$ is*

$$\mathrm{Mon}(\mathcal{P}) = \langle s_0, \ldots, s_{n-1} \rangle$$

(a subgroup of the symmetric group on $\mathcal{F}(\mathcal{P})$).

Remark 2.4. *It is easy to check that* $\mathrm{Mon}(\mathcal{P})$ *is an sggi. Intuitively speaking, a relation* $s_{j_1} \cdots s_{j_m} = 1$ *in this group forces the corresponding type of flag chain to close, regardless of the initial flag in such a chain. This in turn suggests how* $\mathcal{P}$ *arises by identifications in some cover; we refer to [5] for more details. Here we simply quote some easily proved results from [17]. The first of these is verging on folklore.* □

Proposition 2.5. *Let* $\mathcal{R}$ *be a regular* n*-polytope with base flag* Φ*, automorphism group* $\Gamma(\mathcal{R}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$*, and monodromy group* $\mathrm{Mon}(\mathcal{R}) = \langle s_0, \ldots, s_{n-1} \rangle$*. Then there is an isomorphism* $\Gamma(\mathcal{R}) \simeq \mathrm{Mon}(\mathcal{R})$ *mapping each* ρ_j *to* s_j*.*

Proposition 2.6. *Let* $\mathcal{Q}$ *be an* m*-polytope whose vertex-figures are all isomorphic to a particular* $(m-1)$*-polytope* $\mathcal{P}$*; and let* $\mathrm{Mon}(\mathcal{Q}) = \langle t_0, \ldots, t_{m-1} \rangle$*. Then* $\mathrm{Mon}(\mathcal{P}) \simeq \langle t_1, \ldots, t_{m-1} \rangle$*. Furthermore, if* $\mathcal{P}$ *is regular, then* $\mathrm{Mon}(\mathcal{Q})$*is a string C-group.*

Proposition 2.7. *Suppose* $\eta : \mathcal{R} \to \mathcal{P}$ *is a covering of* n*-polytopes (or even pre-polytopes). Then there is an epimorphism*

$$\overline{\eta} : \mathrm{Mon}(\mathcal{R}) \to \mathrm{Mon}(\mathcal{P})$$

(of sggi's, i.e. mapping standard generators to standard generators).

Suppose also that η *maps the flag* Λ' *in* $\mathcal{R}$ *to the flag* Λ *in* $\mathcal{P}$*. Then*

$$(\mathrm{Stab}_{\mathrm{Mon}(\mathcal{R})}\Lambda')\,\overline{\eta} \subseteq \mathrm{Stab}_{\mathrm{Mon}(\mathcal{P})}\Lambda\ . \tag{3}$$

Finally we have a converse to the previous result.

Proposition 2.8. *Suppose that* $\mathcal{R}$ *and* $\mathcal{P}$ *are* n*-polytopes and that*

$$\overline{\eta} : \mathrm{Mon}(\mathcal{R}) \to \mathrm{Mon}(\mathcal{P})$$

is an epimorphism of sggi's. Suppose also that there are flags Λ' *of* $\mathcal{R}$ *and* Λ *of* $\mathcal{P}$ *such that*

$$(\mathrm{Stab}_{\mathrm{Mon}(\mathcal{R})}\Lambda')\,\overline{\eta} \subseteq \mathrm{Stab}_{\mathrm{Mon}(\mathcal{P})}\Lambda\ . \tag{4}$$

Then there is a covering $\eta : \mathcal{R} \to \mathcal{P}$*, which induces* $\overline{\eta}$ *as in Proposition 2.7.*

Remark 2.9. *If* $\mathcal{R}$ *is regular, then condition (3) or (4) is fulfilled automatically, since all flags* Λ' *are equivalent, with trivial stabilizer, in* $\Gamma(\mathcal{R}) \simeq \mathrm{Mon}(\mathcal{R})$ *(see Proposition 2.5). In such cases, a covering* $\eta : \mathcal{R} \to \mathcal{P}$ *induces an epimorphism*

$$\Gamma(\mathcal{R}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle \to \mathrm{Mon}(\mathcal{P}) = \langle t_0, \ldots, t_{n-1} \rangle$$

sending ρ_i *to* t_i*, for* $0 \leqslant i \leqslant n-1$*.* □

3 Geometrical and Combinatorial Versions of Wythoff's Construction

With the above machinery in place, we can take a closer look at the uniform tessellation $\mathcal{U}$ mentioned earlier. As a combinatorial object, this tessellation of $\mathbb{R}^3$ is an abstract uniform 4-polytope (see Figure 1).

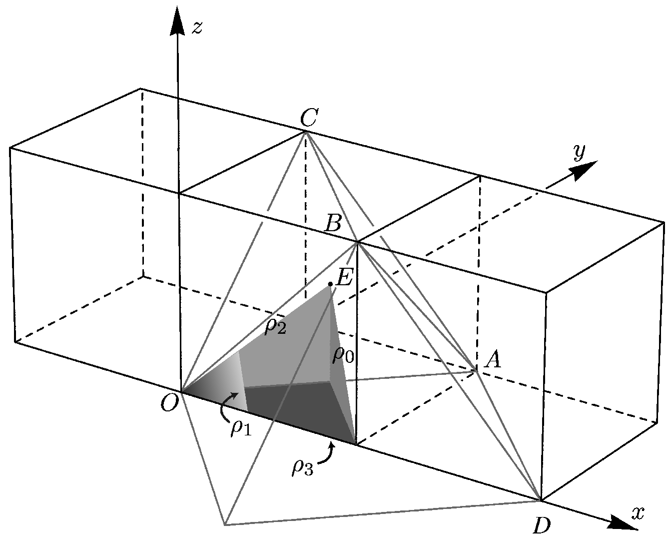


Figure 1: The start of the 4-polytope $\mathcal{U}$, a uniform tessellation of $\mathbb{R}^3$.

A simple way to describe $\mathcal{U}$ is to first imagine Euclidean space $\mathbb{R}^3$ tiled as usual by unit cubes. (Although this tiling is itself a regular 4-polytope, we use it mainly as scaffolding for $\mathcal{U}$.) Each cube has two inscribed regular tetrahedra. Pick one in each cube, starting with the tetrahedron with vertices $O = (0,0,0), A = (1,1,0), B = (1,0,1), C = (0,1,1)$ in the standard unit cube, then alternating thereafter as one passes between adjacent cubes. We thus get the tetrahedral facets of $\mathcal{U}$; the octahedral facets tile what is left of $\mathbb{R}^3$. Every vertex of $\mathcal{U}$ is surrounded by six octahedra and eight tetrahedra; each vertex-figure is a cuboctahedron. (Compare [3, §4.7], where Coxeter describes $\mathcal{U}$ as a *quasiregular* tessellation, with modified Schläfli symbol $\{3, {3 \atop 4}\}$.)

Notice that the symmetry group $\Gamma(\mathcal{U})$ contains the face-centred cubic lattice generated by translations τ_1, τ_2, τ_3 along the edges OA, OB, OC of the base tetrahedron. The point group stabilizing vertex O is the octahedral group of order 48 generated by reflections ρ_1, ρ_2, ρ_3, whose mirrors are indicated in Figure 1, along with the mirror for a fourth reflection ρ_0. These four reflections generate $\Gamma(\mathcal{U})$, and their mirrors enclose a tetrahedral fundamental region for the action of $\Gamma(\mathcal{U})$ on $\mathbb{R}^3$. In fact, $\Gamma(\mathcal{U})$ is the (infinite) Coxeter group of type $\tilde{B}_3$. It is convenient to blur the distinction between the affine reflections ρ_j and their abstract counterparts in a presentation of $\Gamma(\mathcal{U})$, as encoded in the diagram in Figure 2.

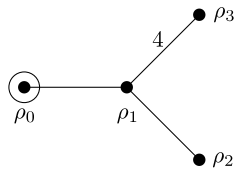


Figure 2: The affine Coxeter group $\tilde{B}_3 \simeq \Gamma(\mathcal{U})$.

Recall [9, §6.5] that $\Gamma(\mathcal{U})$ has these defining relations on its standard generators:

$$\rho_j^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_2\rho_3)^2 = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^3 = (\rho_1\rho_3)^4 = 1\ . \qquad (5)$$

Evidently $\Gamma(\mathcal{U})$ is not a string C-group. Indeed, $\mathcal{U}$ is not regular, for it has two kinds of facets and two flag orbits. Even though $\Gamma(\mathcal{U})$ is not an sggi, it does satisfy (2), as does any Coxeter group [9, Theorem 5.5]. This fact appears in Proposition 3.2 below.

For future use, here we rewrite the generators of the translation subgroup:

$$\tau_1 = \rho_1\rho_3\rho_1\rho_2\rho_1\rho_3\rho_1\rho_0, \quad \tau_2 = \rho_1\tau_1\rho_1, \quad \tau_3 = \rho_2\tau_2\rho_2. \qquad (6)$$

The ringed node in Figure 2 is an ingenious decoration invented by Coxeter [2] and is meant to encode *Wythoff's construction* for $\mathcal{U}$. The essential idea is that each j-face of $\mathcal{U}$ lies in the same $\Gamma(\mathcal{U})$-orbit as a special j-face F_j, whose stabilizer $\Sigma(F_j)$ is a certain parabolic subgroup of $\Gamma(\mathcal{U})$. This $\Sigma(F_j)$ is generated by the ρ_k's corresponding to nodes in a subdiagram, which in turn consists of a connected active part, which has j nodes including the ringed node, and a passive part induced on all nodes not connected to the active part. For example, there is one base vertex $F_0 = O$ in Figure 1; it has empty active part and is fixed (passively) by $\Sigma(F_0) = \langle\rho_1, \rho_2, \rho_3\rangle$. The base edge F_1 has vertices O and $A = (O)\rho_0$; the active part of $\Sigma(F_1) = \langle\rho_0, \rho_2, \rho_3\rangle$ is $\langle\rho_0\rangle$. The base equilateral triangle F_2 has vertices OAB and $\Sigma(F_2) = \langle\rho_0, \rho_1\rangle$ is totally active. Finally, there are octahedral facets F_3 with $\Sigma(F_3) = \langle\rho_0, \rho_1, \rho_3\rangle$, and tetrahedral facets F_3' with $\Sigma(F_3') = \langle\rho_0, \rho_1, \rho_2\rangle$. In this 'geometrical' version of Wythoff's construction, each basic face F_j is the convex hull of the $\Sigma(F_j)$-orbit of O. Observe that general j-faces in the the $\Gamma(\mathcal{U})$-orbit of F_j correspond exactly to right cosets of $\Sigma(F_j)$.

We note that the description above must be adjusted a little to accommodate other sorts of Coxeter diagrams, with arbitrary sets of ringed nodes. However, the regular case works as expected: just ring one terminal node in a string diagram [13, 1B]. We refer to [1] for a much broader look at the concrete geometrical aspects of Wythoff's construction. Here we pursue instead a more 'combinatorial' version of the construction, motivated by our earlier discussion but still tailored to our immediate needs.

Definition 3.1. ***The uniform*** 4***-polytope*** $\mathcal{P} = \mathcal{P}(\Gamma)$ ***and its vertex-figure.***
Suppose $\Gamma = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ *satisfies at least the relations*

$$\rho_j^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_2\rho_3)^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_1\rho_3)^r = 1 \;, \tag{7}$$

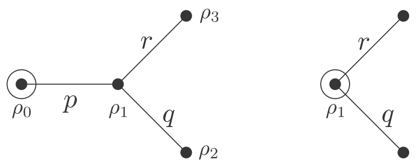
as encoded in Figure 3a. Also suppose that Γ *satisfies the intersection condition (2) for* $I, J \subseteq \{\rho_0, \rho_1, \rho_2, \rho_3\}$*. Subject to verification of details (Proposition 3.2), we define a* 4*-polytope* $\mathcal{P}$ *as follows.*

The improper faces of $\mathcal{P}$ *are two distinguished copies* F_{-1} *and* F_4 *of* Γ*. The proper faces of* $\mathcal{P}$ *are all right cosets of the special subgroups* F_j *defined here:*

Rank j	*Face Type*	*Subgroup*
0	*vertex*	$F_0 := \langle \rho_1, \rho_2, \rho_3 \rangle$
1	*edge*	$F_1 := \langle \rho_0, \rho_2, \rho_3 \rangle$
2	*p-gon* $\{p\}$	$F_2 := \langle \rho_0, \rho_1 \rangle$
3	*facet of Schläfli type* $\{p, q\}$	$F_3 := \langle \rho_0, \rho_1, \rho_2 \rangle$
3	*facet of Schläfli type* $\{p, r\}$	$F_3' := \langle \rho_0, \rho_1, \rho_3 \rangle$

Incidence is defined (for faces of unequal rank) by non-empty intersection of cosets.

Referring to Figure 3b, we define in similar fashion a 3*-polytope* $\mathcal{Q}$ *from the subgroup* $\langle \rho_1, \rho_2, \rho_3 \rangle$*. The base vertex and edge of* $\mathcal{Q}$ *are* $\langle \rho_2, \rho_3 \rangle$ *and* $\langle \rho_1 \rangle$*, respectively. There are two basic* 2*-faces: the* q*-gon corresponding to* $\langle \rho_1, \rho_2 \rangle$ *and the* r*-gon corresponding to* $\langle \rho_1, \rho_3 \rangle$*. Again faces of unequal rank are incident in* $\mathcal{Q}$ *if they have non-empty intersection as cosets in* $\langle \rho_1, \rho_2, \rho_3 \rangle$*.*



(a) The 4-polytope $\mathcal{P}$. (b) The 3-polytope $\mathcal{Q}$.

Figure 3: Wythoff's contruction for the 4-polytope $\mathcal{P}$ and its polyhedral vertex-figure $\mathcal{Q}$.

In fact, $\mathcal{P}$ and $\mathcal{Q}$ really are polytopes. We refer to [16] for a proof of a considerably more general version of Proposition 3.2.

Proposition 3.2. *Suppose* Γ *satisfies the requirements of Definition 3.1. Then* $\mathcal{P} = \mathcal{P}(\Gamma)$ *is an abstract uniform* 4*-polytope with two facets of each type* $\{p, q\}$ *and* $\{p, r\}$ *arranged alternately around each edge.*

Γ *acts faithfully as a group of automorphisms of* $\mathcal{P}$ *and equals the full automorphism group of* $\mathcal{P}$ *if the two base facets are non-isomorphic.*

Each vertex-figure of $\mathcal{P}$ *is a copy of the uniform* 3*-polytope* $\mathcal{Q}$*, which has a pair of* q*-gons and* r*-gons alternating around each of its vertices.*

Remark 3.3. *Certainly, Γ is the full automorphism group for $\mathcal{P}$ when $q \neq r$. However, this can also be so when $q = r$. On the other hand, if the mapping $(\rho_0, \rho_1, \rho_2, \rho_3) \to (\rho_0, \rho_1, \rho_3, \rho_2)$ induces an automorphism of Γ, then $q = r$, and $\mathcal{P}$ is regular of type $\{p, q, 4\}$. In this case, Γ has index 2 in the full automorphism group of $\mathcal{P}$ [16]. Uniform polytopes like these with regular facets are sometimes called semiregular [11].* $\square$

4 The Tomotope $\mathcal{T}$

We can now construct the finite uniform 4-polytopes which are our main concern. Each will be covered by the Euclidean tessellation $\mathcal{U}$. The corresponding automorphism groups are quotients of the Coxeter group $\tilde{B}_3$, obtained by adjoining certain natural relations to those in (5). When applying Proposition 3.2, we therefore take $p = q = 3$ and $r = 4$. In each case we have used *GAP* [4] to verify the intersection condition and other details.

Example 4.1. *Adjoin to (5) the relation which converts the second facet into the hemioctahedron:*

$$(\rho_0\rho_1\rho_3)^3 = 1\ .$$

The resulting group $\Gamma(\mathcal{U}_{t,ho})$ has order 192 but still satisfies the intersection condition. The corresponding polytope $\mathcal{U}_{t,ho}$ is universal for assembling tetrahedra and hemioctahedra in this uniform way, two each alternating around edges. It has 4 vertices, 24 edges, 32 triangles, 8 tetrahedra and 8 hemioctahedra. Each vertex figure is a cuboctahedron. The monodromy group $\mathrm{Mon}(\mathcal{U}_{t,ho})$ has order 73728 and is a string C-group of Schläfli type $\{3, 12, 4\}$. $\square$

Example 4.2. *Adjoin instead the hemicuboctahedron relation*

$$(\rho_2\rho_1\rho_3)^3 = 1$$

to (5) to get a new group $\Gamma(\mathcal{U}_{hc})$. In fact, $\Gamma(\mathcal{U}_{hc})$ is isomorphic to $\Gamma(\mathcal{U}_{t,ho})$, where the latter group has new generators $\rho_0, \rho_1, \rho_2, \rho_0\rho_2\rho_3$. The resulting polytope $\mathcal{U}_{hc}$ has 8 vertices, 24 edges, 32 triangles, 8 tetrahedra and 4 octahedra.

The new monodromy group $\mathrm{Mon}(\mathcal{U}_{hc})$ is a string C-group of type $\{3, 12, 4\}$ and order 73728. However, this group is not isomorphic to $\mathrm{Mon}(\mathcal{U}_{t,ho})$ as an sggi. Indeed, the vertex-figure subgroup for $\mathrm{Mon}(\mathcal{U}_{t,ho})$ has order 2304 and is a double cover of the corresponding subgroup of $\mathrm{Mon}(\mathcal{U}_{hc})$. $\square$

Example 4.3. ***The tomotope.***

Adjoin to (5) the two relations which create hemioctahedra and hemicuboctahedra:

$$(\rho_0\rho_1\rho_3)^3 = (\rho_2\rho_1\rho_3)^3 = 1\ . \tag{8}$$

The new group $\Gamma(\mathcal{T})$ has order 96 and again satisfies the intersection condition.

Definition 4.4. *We shall call the resulting uniform* 4*-polytope* $\mathcal{T}$ *the tomotope.*

The tomotope has 4 vertices, 12 edges, 16 triangles, 4 tetrahedra and 4 hemioctahedra. Each vertex-figure is a hemicuboctahedron. The group $\Gamma(\mathcal{T})$ acts faithfully on the edges of $\mathcal{T}$, so we have this permutation representation:

$$\begin{aligned}
\rho_0 &= (5,10)(6,9)(7,12)(8,11)\\
\rho_1 &= (1,6)(2,5)(3,8)(4,7)\\
\rho_2 &= (5,9)(6,10)(7,11)(8,12)\\
\rho_3 &= (5,8)(6,7)(9,12)(10,11)
\end{aligned}$$

The new monodromy group $\mathrm{Mon}(\mathcal{T}) = \langle s_0, s_1, s_2, s_3\rangle$ has order $18432 = 73728/4$ and again type $\{3, 12, 4\}$. However, the intersection condition on $\mathrm{Mon}(\mathcal{T})$ *fails*. Indeed, for $z = s_2^{s_1 s_2 s_1 s_2 s_1}$ we find that

$$z^{s_0} = z^{s_3 s_2 s_3} \notin \langle s_1, s_2\rangle\ .$$

The earlier monodromy group $\mathrm{Mon}(\mathcal{U}_{t,ho})$ was a string C-group; we may use its central quotient to manufacture a minimal regular cover for $\mathcal{T}$.

Proposition 4.5. *Let* Z *be the centre of* $\mathrm{Mon}(\mathcal{U}_{t,ho})$*. Then* $|Z| = 2$*, and* $\Gamma_2 := \mathrm{Mon}(\mathcal{U}_{t,ho})/Z$ *is a string C-group of order* 36864 *and Schläfli type* $\{3, 12, 4\}$*. The corresponding regular* 4*-polytope* $\mathcal{R}_2$ *has facet type* $\{3, 12\} * 576$ *and vertex-figure type* $\{12, 4\} * 1152b$ *(referring in each case to the census in* [6]*).*

Moreover, $\mathcal{R}_2$ *is a minimal regular cover for the tomotope* $\mathcal{T}$*.*

Proof. Using *GAP* we easily check that there is a $2 : 1$ epimorphism $\Gamma_2 \to \mathrm{Mon}(\mathcal{T})$ (of sggi's). But $\Gamma_2 = \Gamma(\mathcal{R}_2) \simeq \mathrm{Mon}(\mathcal{R}_2)$ by Proposition 2.5. This gives a map $\mathrm{Mon}(\mathcal{R}_2) \to \mathrm{Mon}(\mathcal{T})$, which by Proposition 2.8 induces a covering $\eta : \mathcal{R}_2 \to \mathcal{T}$. On the other hand, any regular cover trapped between $\mathcal{R}_2$ and $\mathcal{T}$ would, by Proposition 2.7, induce a corresponding sequence of maps on the monodromy groups. Since $|\Gamma_2| = 2\cdot|\mathrm{Mon}(\mathcal{T})|$, and $\mathrm{Mon}(\mathcal{T})$ fails the intersection condition, this forces an intermediate 4-polytope to coincide with $\mathcal{R}_2$.

□

4.1 Aside: visualizing the tomotope

To visualize the tomotope $\mathcal{T}$ imagine a core octahedron with 8 tetrahedra glued to its faces, suggesting the *stella octangula*. Next imagine this complex inscribed in a $2 \times 2 \times 2$ cube and from that make toroidal-type identifications. Finally, we further identify antipodal faces of all ranks to get $\mathcal{T}$. In Figure 4 you can see the 4 vertices, $4 = 8/2$ tetrahedra and 1 hemioctahedron in the core. The other three hemioctahedra are red, yellow and green, and 'run around' the belts of those colours. For example, before making identifications, we may split a red octahedron into four sectors around a vertical axis of symmetry, then fit these into four red slots, two of which are visible in Figure 4. In this way we fill out the $2 \times 2 \times 2$ cube before the final antipodal identifications.

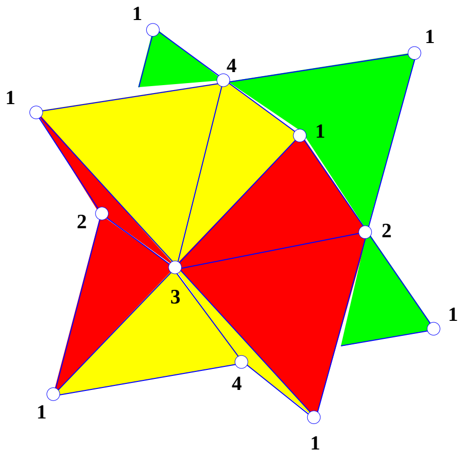


Figure 4: The tomotope $\mathcal{T}$.

5 Toroidal Covers of $\mathcal{T}$

Since the tomotope $\mathcal{T}$ was constructed as a quotient of the uniform Euclidean tessellation $\mathcal{U}$, it is natural to ask whether other quotients of $\mathcal{U}$ provide covers of $\mathcal{T}$.

Proposition 5.1. *Let $\mathcal{U}$ be the uniform tessellation of $\mathbb{R}^3$ described above; and let $\Gamma = \Gamma(\mathcal{U})$ be its symmetry group (see Figure 2). Suppose Σ is a subgroup of the translation group $\langle \tau_1, \tau_2, \tau_3 \rangle$ of Γ, such that each orbit of Σ meets each proper section of $\mathcal{U}$ in at most one face.*

Then the quotient $\mathcal{Q} = \mathcal{Q}_\Sigma := \mathcal{U}/\Sigma$ is a (toroidal) 4-polytope, still with regular octahedral and tetrahedral facets, two each alternating around each edge.

Furthermore, the monodromy group $\mathrm{Mon}(\mathcal{Q})$ is a string C-group of type $\{3, 12, 4\}$.

Proof. It is clear on geometrical grounds that $\mathcal{Q}$ really is a polytope with the indicated properties, although we should use the modifier 'toroidal' only when the generators of Σ span $\mathbb{R}^3$. For a more rigorous proof we can refer to Proposition 2D9 in [13]; the indicated restrictions on Σ are exactly what is needed there.

It remains to verify the intersection condition for $\mathrm{Mon}(\mathcal{Q}) = \langle t_0, t_1, t_2, t_3 \rangle$. From Proposition 2.6, we have $\langle t_1, t_2, t_3 \rangle \simeq \mathrm{Mon}(\mathcal{P})$, where the cuboctahedron $\mathcal{P}$ is isomorphic to the vertex-figure at each vertex of $\mathcal{Q}$. Applying Proposition 2.6 again to $\mathcal{P}$ itself, where every vertex-figure is a 4-gon, we find that $\langle t_1, t_2, t_3 \rangle$ is a string C-group of type $\{12, 4\}$ and order 2304. The corresponding regular 3-polytope $\tilde{\mathcal{P}}$ is the minimal regular cover of $\mathcal{P}$ [7]. The cuboctahedron is, of course, orientable when considered as a spherical map; the triangles in a barycentric subdivision of the surface can be coloured alternately black and white. It follows that any relation $t_{j_1} \cdots t_{j_m} = 1$ in the generators t_1, t_2, t_3 must have even length. The words of even length therefore constitute a subgroup of index 2 in $\langle t_1, t_2, t_3\rangle$; let us call such elements 'even' and those in the other coset 'odd'. Note that $\tilde{\mathcal{P}}$ is itself orientable.

To prove that $\mathrm{Mon}(\mathcal{Q})$ is a string C-group we need only show for $k = 0, 1, 2$ that $g \in \langle t_0, \ldots, t_k\rangle \cap \langle t_1, t_2, t_3\rangle$ implies $g \in \langle t_1, \ldots, t_k\rangle$ [13, 2E16(b)]. For such g and any flag Φ of $\mathcal{Q}$, both Φ and Φ^g have the same proper faces of rank $j \in \{0, k+1, \ldots, 3\}$. If $k = 0$ this automatically means $\Phi^g = \Phi$ for all flags Φ, so $g = 1$.

Suppose then that $k = 2$ and fix a particular flag Φ, say with an octahedral facet. Without loss of generality we may assume that g is an even element of $\langle t_1, t_2, t_3\rangle$; otherwise we could could take gt_1 instead. Since $\mathcal{Q}$ is strongly flag-connected, we have $\Phi^g = \Phi^h$ for some $h \in \langle t_1, t_2\rangle$. Moreover, since g can be written as an even word in $\{t_1, t_2, t_3\}$, we have that $h = (t_1t_2)^m$ for some $m \in \{0, 1, 2, 3\}$. (Note: since $g \in \langle t_1, t_2, t_3\rangle$, g fixes the vertex in any flag of $\mathcal{Q}$, and in particular must act like a rotation around the vertex of the octahedron.)

Let Ψ be another flag of $\mathcal{Q}$ whose facet is an octahedron and let η be an isomorphism between the octahedral facet of Φ to that of Ψ. Then

$$\Psi^g = (\Phi\eta)^g = \Phi^g\eta = \Phi^h\eta = (\Phi\eta)^h = \Psi^h.$$

Thus g acts like $(t_1t_2)^m$ on *all* flags with an octahedral facet. A similar argument will show that there exists $m' \in \{0, 1, 2\}$ such that for each flag Λ whose facet is a tetrahedron, we have $\Lambda^g = \Lambda^{m'}$, where $h' = (t_1t_2)^{m'}$.

Then $g = (t_1t_2)^l \in \langle t_1, t_2\rangle$ on *all* flags of $\mathcal{Q}$, where $l = 4m' - 3m$. (We use the fact that $\gcd(4, 3) = 1$.) The remaining case $k = 1$ is similar and easier. $\square$

Remark 5.2. *The subgroup $\langle t_0, t_1, t_2\rangle$ is also a string C-group and has type $\{3, 12\}$. Taking $\Gamma(\{3, 3\}) = \langle \alpha_0, \alpha_1, \alpha_2\rangle$ and $\Gamma(\{3, 4\}) = \langle \beta_0, \beta_1, \beta_2\rangle$, we find that the mapping*

$$\begin{array}{rcl} \langle t_0, t_1, t_2\rangle & \to & \Gamma(\{3,3\}) \times \Gamma(\{3,4\}) \\ t_j & \mapsto & (\alpha_j, \beta_j) \end{array}$$

is a well-defined injection. The image group, which we denote $\Gamma(\{3,3\})\Diamond\Gamma(\{3,4\})$, is called the mix, or parallel product, of $\Gamma(\{3,3\})$ and $\Gamma(\{3,4\})$; see [21, 18] or [17]. It has order $576 = 24^2$ and consists of all (α, β) such that α, β are either both even or both odd.

We note that the full translation subgroup of $\Gamma(\mathcal{U}) = \tilde{B}_3$ can be identified with the group of all integer vectors $\langle x, y, z\rangle$ such that $x + y + z$ is even. The condition on the subgroup Σ in Proposition 5.1 is equivalent to demanding that Σ contain none of the vectors $\langle \pm1, \pm1, 0\rangle, \langle \pm2, 0, 0\rangle$ for any choice of sign or permutation of coordinates. In other words, the ball $x^2 + y^2 + z^2 \leqslant 4$ should meet Σ only at $\langle 0, 0, 0\rangle$. $\square$

Referring to Figure 1, we see that the translation $\tau_1\tau_2\tau_3^{-1}$ has vector $\overrightarrow{OD} = \langle 2, 0, 0\rangle$. For each integer $k \geqslant 1$, the relation

$$(\tau_1\tau_2\tau_3^{-1})^k = 1 \tag{9}$$

and its conjugates in Γ effectively induce toroidal identifications on the opposite faces of a $2k \times 2k \times 2k$ block (of unit cubes). This family of toroids will be very useful to us:

Definition 5.3. *For $k \geqslant 1$, let $\Sigma(k)$ be the normal subgroup of Γ generated by $(\tau_1\tau_2\tau_3^{-1})^k$. Let $Q_k := \mathcal{Q}_{\Sigma(k)}$ be the corresponding uniform toroid, as described in Theorem 5.1. Let $W_k := \Gamma/\Sigma(k)$ be the corresponding finite quotient group.*

Lemma 5.4. *For $k \geqslant 1$, the group W_k is presented by adjoining the extra relation (9) to those in (5), and*

$$|W_k| = 24 \cdot (2k)^3 \ .$$

For $k > 1$, the polytope $\mathcal{Q}_k$ has $4k^3$ vertices, $24k^3$ edges, $32k^3$ triangles, $8k^3$ tetrahedra and $4k^3$ octahedra; and $\mathrm{Mon}(Q_k)$ is a string C-group. When $k = 1$, $\mathcal{Q}_1$ is merely a pre-polytope and $\mathrm{Mon}(\mathcal{Q}_1)$ is not a string C-group. Moreover, $|\mathrm{Mon}(\mathcal{Q}_1)| = 36864$, and there is a $2:1$ epimorphism

$$\mathrm{Mon}(\mathcal{Q}_1) \to \mathrm{Mon}(\mathcal{T})$$

(of sggi's).

Proof. Since $\tilde{B}_3 \simeq \Gamma$, we may as well reason geometrically. The toroid Q_k contains $(2k)^3$ unit cubes, each of which contains $24 = 48/2$ copies of the fundamental simplicial domain for $\tilde{B}_3$. Each unit cube contains exactly one tetrahedron, as well as half an octahedron.

The details concerning $\mathrm{Mon}(\mathcal{Q}_k)$ follow at once from Theorem 5.1, when $k > 1$, and by explicit calculation in *GAP* when $k = 1$. □

Remark 5.5. *With the help of Figure 1, it is easy to check that $\Sigma(k)$ is generated by $(\tau_1\tau_2\tau_3^{-1})^k$, $(\tau_2\tau_3\tau_1^{-1})^k$ and $(\tau_3\tau_1\tau_2^{-1})^k$.*

The above construction of Q_k is essentially geometric. On the other hand, when $k > 1$, the group W_k, with the four induced involutory generators, satisfies (2) and hence the requirements of Proposition 3.2 (replacing Γ by W_k). The resulting abstract regular polytope is, of course, isomorphic to Q_k. □

Definition 5.6. *For $k > 1$, let $\mathcal{P}_k$ be the regular 4-polytope whose automorphism group is $\mathrm{Mon}(\mathcal{Q}_k)$.*

In order to connect the regular polytopes $\mathcal{P}_k$ to the tomotope $\mathcal{T}$, we shall exploit an element in the monodromy group, which vividly illustrates the difference between monodromy and symmetry.

Definition 5.7. *In any sggi $\langle t_0, t_1, t_2, t_3 \rangle$, let*

$$w = t_0(t_1t_2)^6 t_0t_1t_3t_2t_3(t_1t_2)^6 t_3t_2t_3t_1(t_1t_2)^6 \ .$$

It will be convenient to abuse notation by simultaneously letting w refer to this element in several sggi's of rank 4.

Lemma 5.8. (a) *In* $\mathrm{Mon}(Q_k) = \langle t_0, t_1, t_2, t_3\rangle$, *the element* w *fixes each flag containing a tetrahedral facet. On the other hand, if* Φ *is a flag with an octahedral facet, then* w *shifts* Φ *in exactly the same way as the translation from the vertex* X *of* Φ *to the opposite vertex* Y *in that equatorial square of the octahedron which contains the edge of* Φ.

(b) *The element* w *has period* k *in* $\mathrm{Mon}(\mathcal{Q}_k)$. *In particular,* $w = 1$ *in* $\mathrm{Mon}(\mathcal{Q}_1)$.

(c) *There is an epimorphism* $\overline{\eta} : \mathrm{Mon}(\mathcal{Q}_k) \to \mathrm{Mon}(\mathcal{Q}_1)$ *of sggi's whose kernel is the elementary abelian group of order* k^6 *generated by*

$$w, t_0wt_0, t_1t_0wt_0t_1, t_3wt_3, t_3t_0wt_0t_3, t_3t_1t_0wt_0t_1t_3\ . \tag{10}$$

(d) *The group* $\mathrm{Mon}(\mathcal{Q}_k)$ *has order* $576 \cdot (2k)^6 = 36864 \cdot k^6$.

Proof. Ponder a model or Figure 1. Notice that w 'disconnects' the octahedral flags and independently maps them in the three orthogonal directions parallel to the diagonals of an octahedron. Now, in the group $\mathrm{Mon}(\mathcal{Q}_k)$, translations still commute and the basic translation $\overrightarrow{OD}$ from Figure 1 has period k. Thus the elements in (10) commute, have period k and generate a subgroup N of order k^6 in $\mathrm{Mon}(\mathcal{Q}_k)$.

It is clear from our construction that $\mathcal{Q}_k$ covers $\mathcal{Q}_m$ whenever m divides k. Proposition 2.7 then gives an epimorphism $\mathrm{Mon}(\mathcal{Q}_k) \to \mathrm{Mon}(\mathcal{Q}_m)$.

For $\overline{\eta} : \mathrm{Mon}(\mathcal{Q}_k) \to \mathrm{Mon}(\mathcal{Q}_1)$, we observe that $N \subseteq \ker(\overline{\eta})$. To show equality we take $f \in \ker(\overline{\eta})$ and fix a hemioctahedral flag Φ. Since $f \in \ker(\overline{\eta})$, Φ is fixed modulo the action of the translation subgroup $\Sigma(k)$ on $\mathcal{Q}_k$. Thus there exists $\tau \in \Sigma(k)$ such that $\Phi^f = \Phi\tau$. But $\Phi\tau = \Phi^h$ for some $h \in \langle w, t_0wt_0, t_1t_0wt_0t_1\rangle$, so $\Phi^{fh^{-1}} = \Phi$. Notice that we can and do choose an h which fixes all tetrahedral flags Λ'.

But the action of $\mathrm{Mon}(\mathcal{Q}_k)$ on flags commutes with that of the automorphism group W_k, so for any hemioctahedral flag $\Phi' = \Phi\gamma$, $\gamma \in W_k$, we have $(\Phi')^{fh^{-1}} = \Phi'$. Similarly there exists $g \in \langle t_3wt_3, t_3t_0wt_0t_3, t_3t_1t_0wt_0t_1t_3\rangle$ such that $(\Lambda')^{fg^{-1}} = \Lambda'$ for all tetrahedral flags Λ'. Now g fixes all hemioctahedral flags.

But $gh = hg$, so that $fh^{-1}g^{-1} = 1$ in $\mathrm{Mon}(\mathcal{Q}_k)$ and $f = gh \in N$. This proves that $N = \ker(\overline{\eta})$ has order k^6. Part (d) follows from Lemma 5.4. $\square$

Theorem 5.9. *Let* $p, q > 1$ *be coprime odd integers. Then the tomotope* $\mathcal{T}$ *has minimal regular covers* $\mathcal{R}_p$ *and* $\mathcal{R}_q$, *neither of which covers the other or the regular cover* $\mathcal{R}_2$ *from Proposition 4.5.*

Proof. As in the proof of Lemma 5.8, we have $\mathrm{Mon}(\mathcal{Q}_p) \to \mathrm{Mon}(\mathcal{Q}_1) \to \mathrm{Mon}(\mathcal{T})$. The string C-group $\mathrm{Mon}(Q_p) \simeq \Gamma(\mathcal{P}_p)$ therefore provides a regular cover $\mathcal{P}_p$ for $\mathcal{T}$, by Proposition 2.8. Hence there must exist some *minimal* regular cover, say $\mathcal{R}_p$, satisfying $\mathcal{P}_p \to \mathcal{R}_p \to \mathcal{T}$. Likewise we have a minimal regular cover $\mathcal{R}_q$.

Now suppose $\mathcal{R}_p$ and $\mathcal{R}_q$ cover some regular polytope $\mathcal{R}$, which in turn covers $\mathcal{T}$. Then $\mathrm{Mon}(\mathcal{P}_p)$ and $\mathrm{Mon}(\mathcal{P}_q)$ both cover $\mathrm{Mon}(\mathcal{R}) \simeq \Gamma(\mathcal{R})$; since

$\gcd(p,q) = 1$, it follows from Lemma 5.8 that the special element $w = 1$ in $\Gamma(\mathcal{R})$. From Definition 5.7 we have that

$$t_3t_2t_3(t_1t_2)^6t_3t_2t_3 = t_1t_0(t_2t_1)^6t_0(t_2t_1)^6t_1$$

holds in the string C-group $\Gamma(\mathcal{R})$. Since the intersection condition holds here, we must have

$$t_1t_0(t_2t_1)^6t_0(t_2t_1)^6t_1 \in \langle t_0, t_1, t_2\rangle \cap \langle t_1, t_2, t_3\rangle = \langle t_1, t_2\rangle \ .$$

Mapping now to $\mathrm{Mon}(\mathcal{T})$ we conclude that

$$z := t_0(t_2t_1)^6t_0 \in \langle t_1, t_2\rangle \ .$$

But this is false by direct calculation. Indeed, z fixes all tetrahedral flags in $\mathcal{T}$, yet acts like the reflection $t_1t_2t_1$ on any hemioctahedral flag. This is a contradiction, since an element of the dihedral group $\langle t_1, t_2\rangle$ cannot act on different flags of the dodecagon $\{12\}$ in both even and odd ways.

Certainly $\mathcal{R}_p$ cannot cover $\mathcal{R}_q$ or vice versa. For similar reasons, neither of these regular polytopes can cover $\mathcal{R}_2$, since $\gcd(2,p) = 1$ but $w \neq 1$ in $\Gamma_2 = \Gamma(\mathcal{R}_2)$. □

It is clear now that the tomotope must have infinitely many distinct minimal regular covers $\mathcal{R}_p$. In fact, it seems likely from the computational evidence that $\mathcal{R}_p = \mathcal{P}_p$, in other words, the regular cover arising from Definition 5.6 is actually minimal. But we will leave the investigation here.

Surely this sort of multiplicity of minimal regular covers for the tomotope is typical of polytopes having rank at least 4. The uniqueness of such covers for polytopes of lower rank is welcome but atypical.

5.1 Another aside: a brief detour into crystallography

We have observed that the Coxeter group $\tilde{B}_3 \simeq \Gamma(\mathcal{U})$ leaves invariant the face-centred cubic lattice and so is rightly called *crystallographic*. This enables a faithful representation in $GL_4(\mathbb{Z})$ in which the generators appear as

$$\rho_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \rho_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

$$\rho_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \rho_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(see [14] or [9, §6.6]).

Now reduce the infinite group $\tilde{B}_3$ modulo an integer $r \geqslant 2$ and call the resulting finite group M_r. Since the new generators still have period 2, we keep calling them ρ_j. Once one has the basic translations τ_j under control, it is easy to explicitly calculate group orders and other details. For odd $k > 1$, we find that the earlier group $W_k \simeq M_{2k}$; and for even $k > 1$, W_k is a $2:1$ cover of M_{2k}.

The most pertinent case, however, is $r = 2$. (Now the ρ_j are *transvections* rather than reflections.) We find that M_2 has order 96 and is a semidirect product $\mathbb{Z}_2^4 \rtimes \mathbb{S}_3$. In fact, reduction modulo 2 kills most of the translation subgroup of $\tilde{B}_3$ and collapses the periods of $\rho_0\rho_1\rho_3$ and $\rho_2\rho_1\rho_3$ from 6 to 3. Therefore $M_2 \simeq \Gamma(\mathcal{T})$, and we have reconstructed the automorphism group of the tomotope in another natural way.

6 Reye's Configuration

The incidence information for the faces of the tomotope is summarized in Figure 5. The face-numbers for ranks $j = -1, 0, 1, 2, 3, 4$ are indicated inside the circular nodes, reading left to right. The subscripts give incidence numbers for faces of adjacent rank. For example, there are 12 edges and 4 vertices; each edge lies on 2 vertices and each vertex on 6 edges (one for each 0-face of the hemicuboctahedron). As a simple check, we note that $4 \cdot 6 = 12 \cdot 2$.

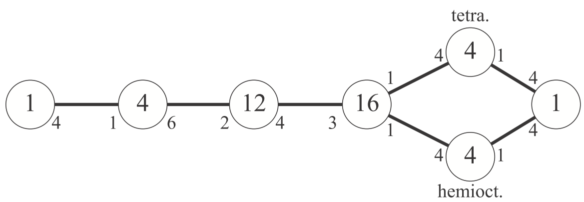


Figure 5: Incidence diagram for the tomotope $\mathcal{T}$.

To understand the finer detail of the incidence structure, we envision the Hasse diagram for $\mathcal{T}$ as a graph and take the subgraph $I_{j,j+1}$ on faces of ranks j and $j+1$, for $-1 \leqslant j \leqslant 3$. Trivially, $I_{-1,0} = K_{1,4}$ (a complete bipartite graph), whereas $I_{3,4} = K_{4+4,1}$. For $j = 0$, we note that each vertex of $\mathcal{T}$ is joined by two edges to each other vertex, so that the 1-skeleton of $\mathcal{T}$ could be denoted $K_4^{(2)}$; thus $I_{0,1}$ is the subdivision graph of $K_4^{(2)}$. Since every triangle lies between a tetrahedron and hemioctahedron, we see that $I_{2,3}$ is likewise the subdivision graph of $K_{4,4}$. For $j = -1, 0, 2$ or 3, it is actually easy to reconstruct the relevant portion of the Hasse diagram from $I_{j,j+1}$.

It was Tomo who directed our attention to the really interesting case $I_{1,2}$. Following [15] we can call $I_{1,2}$ the *medial layer graph* for $\mathcal{T}$. Tomo noticed that the configuration parameters $(12_4, 16_3)$ are exactly those for the well-known *Reye's configuration.* Is there a connection?

Recall that Reye's configuration $\mathcal{K}$ involves lines and points in ordinary real projective space. (See [19] for pictures and related obstructions to drawing $\mathcal{K}$ in the plane.) We may take the 16 lines of $\mathcal{K}$ to lie along the 12 edges and 4 diagonals of a cube; the 12 points of $\mathcal{K}$ are the vertices and centre of the cube, together with the 3 points at infinity from the parallel classes of edges of the cube. Each point therefore lies on 4 lines, each line on 3 points.

To see that the edges and triangles of the tomotope are arranged in exactly the same way, we need some notation. Label the hemioctahedra $1, 2, 3, 4$ and the tetrahedra $-1, -2, -3, -4$. The triangular faces of $\mathcal{T}$ can then be identified with the 16 ordered pairs $(a, -b)$, with $a, b \in J := \{1, 2, 3, 4\}$. After inspecting a model or Figure 4, we find that we can label the edges of $\mathcal{T}$ by the symbols Xe, where X is a 2-subset of J and $e = +$ or $-$. An edge $X-$ is incident with the four triangles in $X \times (-X)$; an edge $X+$ is incident with the triangles in $X \times (-X^c)$, where X^c is the complement of X in J. It is easy to check that triangle $(a, -b)$ really is incident with three edges, all with $e = -$ when $a = b$, otherwise one with $e = -$ and two with $e = +$ when $a \neq b$.

With a little patience, one can label the points and lines of $\mathcal{K}$ with the same symbols and according to the same rules. This gives us most of

Proposition 6.1. *The medial layer graph $I_{1,2}$ for $\mathcal{T}$ is the Levi graph for Reye's configuration $\mathcal{K}$. The automorphism group K for $I_{1,2}$ has order 576 and is a split extension of the tomotope group $\Gamma(\mathcal{T})$ by a dihedral group of order 6.*

Proof. We compute the group K using *GAP* and the subsidiary packages *GRAPE* and *naughty* [20, 12]. Since $\Gamma(\mathcal{T})$ acts faithfully on edges of $\mathcal{T}$, it must embed into K. $\square$

Remark 6.2. *Recall that the Levi graph for a point-line configuration is just the incidence graph on the elements of the configuration. The automorphism group of this graph is sometimes called the group of the configuration itself. It would of course be nice to have a more insightful explanation for why the tomotope contains Reye's configuration.*

The group K is not isomorphic to the facet group of type $\{3, 12\}$ and order 576 which appears in the monodromy group $\mathrm{Mon}(\mathcal{T})$. $\square$

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