GEOMETRY IN A NUTSHELL

NOTES FOR MATH 3063

B. R. MONSON
DEPARTMENT OF MATHEMATICS & STATISTICS
UNIVERSITY OF NEW BRUNSWICK

One morning an acorn awoke
beneath its mother and declared,
“Gee, I’m a tree”.

Anon. (deservedly)

THE TREE OF EUCLIDEAN GEOMETRY

INTRODUCTION

Mathematics is a vast, rich and strange subject. Indeed, it is so varied that it is considerably more difficult to define than say chemistry, economics or psychology. Every individual of that strange species *mathematician* has a favorite description for his or her craft. Mine is that *mathematics is the search for the patterns hidden in the ideas of space and number* .

This description is particularly apt for that rich and beautiful branch of mathematics called geometry. In fact, geometrical ideas and ways of thinking are crucial in many other branches of mathematics.

One of the goals of these notes is to convince you that this search for pattern is continuing and thriving all the time, that mathematics is in some sense a living thing. At this very moment, mathematicians all over the world[1] are discovering new and enchanting things, exploring new realms of the imagination. This thought is easily forgotten in the dreary routine of attending classes.

Like other mathematical creatures, geometry has many faces. Let's look at some of these and, along the way, consider some advice about doing mathematics in general.

Geometry is learned by doing: Ultimately, no one can really teach you mathematics — you must learn by doing it yourself. Naturally, your professor will show the way, give guidance (and also set a blistering pace). But in the end, you will truly acquire a mathematical skill only by working through things yourself.

By the way, it is certainly fine to work in a group (i.e. with one or more friends), if that suits you better. But always be very certain that you yourself understand everything that the group has done.

One essential way to become involved in the course is to try many of the problems, assigned or not. You can learn quite a lot and get great satisfaction from solving tricky problems. Good assignments can be thought provoking and can help you learn by doing.

You can learn even more by conducting your own mathematical experiments. For instance, at several points in this course you will benefit by making mathematical models, say from paper, cardboard or other materials. When you do such activities, be neat, precise and careful; try to understand what you see. A good test of your understanding is this: can you explain what is going on to some friend in residence, your neighbour or even your professor? If so, you have definitely learned something.

In some ways, mathematics is an experimental science. Mathematicians seldom discover new things in the manner in which they are typically portrayed in textbooks. Instead, they make models, conduct numerical experiments (by hand or by computer), play games or design 'thought experiments'.

Geometry is a logical art: In mathematics, we don't just look for patterns—we

[1]For your information, there are over 50,000 individuals listed in the directory of the major American professional societies for mathematicians.

try to explain them in a logical way, and thereby achieve a deeper understanding. Only through this effort can we discover new things or solve harder problems.

For example, we might observe that in various triangles the three angles sum to 180°. Why? (We shall soon see the answer; but, in fact, there are useful and marvelous *geometries* where the angles sum to something else!)

Or a decorative artist might find that there seem to be only seven mathematically distinct ways to decorate a strip of ribbon. Why? (This is a much harder question—see Section 22.)

You have likely had very little experience with mathematics as a logically growing organism. In this course, you will attain just enough familiarity with the logical development of geometry to appreciate the new material later in the course. Throughout this project, we must keep a few things in mind.

Reading a mathematical text written in traditional theorem–proof style is by no means the only or easiest way to learn the material. And the material itself was doubtless originally discovered in a haphazard way, then reworked several times into some final, more 'elegant' form.[2] A good analogy is the way that a poet may jot down some rough thoughts, then work over them several times till some satisfactory verse appears.

On the other hand, there is much pleasure and understanding to be gained from the effort required in following the logical growth of a mathematical subject. So for a few weeks we will indulge in this, in a fairly gentle way.

One warning is due: geometry concerns the basic structures of our space and perception, so it stands to reason that the early stages of its logical development involve things that hardly seem to need proof. Thus, you may well ask, 'Why are we proving the obvious?'. Be patient — we shall try to answer this very reasonable question. And soon enough, we will encounter some beautiful things that are not at all obvious.

Geometry has its own language: One difficulty in coping with any mathematical subject is the use of perfectly ordinary words in the strangest way. Yet good definitions lend precision and economy to mathematical conversation. For the sake of understanding the material, you must carefully learn and use our mathematical vocabulary. In mathematics, creativity and precision live together.

[2] This is true of these notes, too.

$\triangle\triangle$ $\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\iota\kappa\lambda\mu\nu\xi o\pi\rho\sigma\tau\upsilon\phi\chi\psi\omega$ $\triangle\triangle$

I have used these notes, in one form or another, in several attempts at teaching Math 3063. The notes are not at all static— frequently I adjust them to suit my own changing interests, and of course to correct occasional errors. So what you have here is the version which exists on February 7, 2005. If you do find errors or can think of improvements, let me know!

I wish to thank Eleanor Perrin and Linda Guthrie, of the Department of Mathematics and Statistics, for typing in LaTeX the electronic version of this manuscript. Thanks are due as well to Rita Monson for rendering many of the figures in `xfig`, thus allowing me much more flexibility in revising the text.

1 A Short History of Geometry

The word *geometry* derives from the ancient Greek meaning 'measurement of the earth'. Yet despite such ancient roots, geometry is a modern and thriving mathematical science. Indeed, various basic ideas were known to several old civilizations; however, as a mathematical discipline, geometry flourished especially in Greece, Babylon and Egypt some 2000 to 3000 years ago.

Like all mathematical sciences geometry has both an *inductive* and a *deductive* side. When we assemble facts in an *inductive* manner, we first observe some natural phenomena, then try to explain the facts by some general rule or theory. For instance, in ancient Egypt the yearly inundation of the Nile wiped out the boundaries of landowners; thus simple geometrical techniques were required by surveyors to re-establish property lines. It was known, for instance, that a knotted rope with 12 equal segments would form a right angle (an angle of 90°)— see Figure 1.

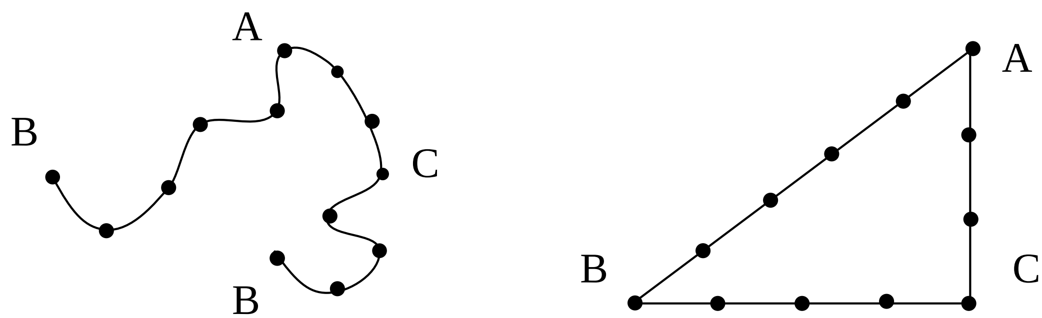


Figure 1: Knotted ropes.

This fact, which *we* know to be true by Pythagoras' theorem, since $5^2 = 3^2 + 4^2$, was initially just a natural observation.[3] There was no general theorem for right triangles (Pythagoras wasn't yet born!) and there certainly was no 'proof' that $\angle ACB = 90°$. Thus with time a large but unsystematic body of geometrical facts was assembled by surveyors, astronomers, navigators and other observers of nature.

The period 600 B.C. - 200 B.C. saw the flowering of a Greek civilization which was in many ways unique. There was a deep and often mystical respect for philosophy, art, mathematics and science. One of the major intellectual accomplishments of ancient Greece was to organize and reduce to basic principles the accummulated mass of geometrical facts. Many Greeks contributed to this effort; we mention only Pythagoras (ca. 548-495 B.C.), Hippocrates (ca. 400 B.C.), Eudoxus (408-355 B.C.), Archimedes (ca. 287-214 B.C., and the greatest of the ancient mathematicians) and Appolonius (ca. 260-170 B.C.). Most familiar of all however is Euclid of Alexandria, who lived about 300 B.C.. Indeed, the ordinary practical geometry of our world is called *Euclidean geometry*, although few of the actual results in geometry are actually Euclid's inventions. Instead, it was his genius to isolate the crucial concepts in geometry and from them deduce, in a logical manner, other results of

[3] Historians of science disagree on just how much the Babylonians and Egyptians knew of this famous theorem and whether Pythagoras was even the first to prove it. You should treat all statements in the history of mathematics with some skepticism.

greater and greater complexity and beauty. In short, Euclid developed the *deductive* side of geometry.

Euclid assembled his geometrical results into a series of 13 short books called the *Elements*.[4] This is perhaps the most widely published textbook ever, though it has, of course, been adapted many times to suit the tastes of scholars and educators over the years. In fact, any high school geometry course is based ultimately on the Elements. The authoritative version in English is listed as [8] in the References. In this set of 3 volumes, Euclid's actual text is fairly short but is accompanied by a very detailed (sometimes dry, but often interesting) commentary by Sir T. L. Heath.

Often geometrical theorems are referred to by their place in the Elements. Thus Pythagoras' theorem is Euclid I - 47, which refers to the 47th result in Book I. Here is a rough table of contents for the thirteen Books of the Elements:

Book	Subject
I	triangles
II	rectangles
III	circles
IV	polygons
V	proportion
VI	similar figures
VII-X	number theory - prime and perfect numbers
XI	geometry of space
XII	pyramids, cones, cylinders
XIII	regular solids - cube, tetrahedron, octahedron, dodecahedron, icosahedron.

[4]The word 'book' is somewhat misleading - most likely papyrus scrolls were used. It seems that the Elements were written as general preparation for studies in philosophy, music, astronomy. Only a very elite and special group of people could have been concerned with such studies. There were then no professional mathematicians or scientists as we understand it. Of course, merchants, sailors and artisans would have mastered the practical side of their crafts. But these were a very different group of people. Our view of geometry and mathematics as a practical tool for the sciences is quite modern.

2 The Deductive Method

In the logical development of geometry (or calculus or other branches of mathematics), each *definition* of a *concept* involves other concepts or relations. Thus, the only way to avoid a vicious circle is to accept certain *primitive* concepts and relations as undefined. Likewise, the proof of each proposition (or theorem) uses other propositions; and hence to again avoid a vicious circle, we must accept certain primitive propositions - called *axioms* or *postulates* - as true but unproved. (I have here paraphrased a particularly apt description taken from reference [3, pages 4 – 5].)

Example. Dictionaries don't worry about vicious circles. From mine, here are some words and definitions:

Word	Definition
Love:	to hold dear: $\underbrace{\text{cherish}}$
Cherish:	to feel or show $\underbrace{\text{affection}}$ for
Affection:	$\underbrace{\text{tender}}$ attachment: fondness
Tender:	fond; $\underbrace{\text{loving}}_{\text{!!!}}$

Thus we encounter a vicious circle after four steps, since in order to define 'love', we ultimately must use the word 'love' itself. Similarly, in logic or mathematics, a common but deadly sin is circular reasoning, in which we ultimately assume what we want to prove.

In a nutshell, Euclid took certain primitive ideas, which everyone would be willing to believe, and used them in a logical way as building blocks for more and more complicated results. This is the essence of the *deductive* method in mathematics. The force of the method is that if you believe the axioms (which are usually 'obviously' true), then you *must* believe the theorems which follow (no matter how far-fetched).

It is a wonderful fact indeed that many *unsuspected* results can be proved on the basis of a certain number of 'obvious' assumptions. Of course, these axioms must not contradict one another. And if there are fewer axioms rather than very many, we should be better able to understand what makes our mathematics work.

Even so, you still might ask, "Why prove anything - why not assume everything?". The answer is that many useful facts are not at all obvious. I'm willing to bet that you wouldn't guess Pythagoras' theorem without any hints; however, you might well invent a proof or even discover the theorem in a mathematical way, that is, by working from simple to more complicated results in a logical manner. In fact, we will do this in a later section. To get some idea how basic concepts lead to more intricate situations, study the tree of Euclidean geometry on page 3.

In summary, we shall isolate the truly basic structures of the geometrical world, ignoring its less important properties. We shall then build on this using the deductive method. We might, if we are lucky, discover results which we would never otherwise guess. After a while, we shall employ our mathematical tools to examine structures and patterns in art and nature.

3 The Building Blocks

3.1 Choosing Axioms

We must start somewhere if we are to build up geometry in a logical manner; unfortunately, however, it's not at all clear where to begin. Euclid made an excellent attempt, although (not surprisingly) modern mathematicians - such as Pasch, Peano, and Hilbert in the 19th century - have found and corrected several of Euclid's oversights. Indeed, if we permit no unstated assumptions, then we must have axioms which justify even obvious statements such as this:

"If $AB = CD$ and $CD = EF$, then $AB = EF$"

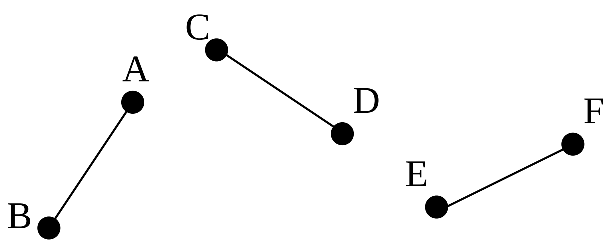


Figure 2: Congruent segments.

The modern list of axioms for geometry is thus long and rather difficult to work with, since it concerns such basic ideas. Don't worry! We shall cheat by simply putting aside and ignoring some of the more common sense ideas. Let's lump together these basic things and call them *foundations*.

It's more important to point out and select the really crucial building blocks as *axioms*. We then use brainpower to glue these blocks together, thereby constructing bigger and more exotic structures (called *theorems*). Later on, we shall require a new type of building block, since some structures just cannot be built from our supply of simple bricks! Finally, when our edifice is nearly complete we will be better able to look back and see what sort of assumptions are hidden in the foundations.

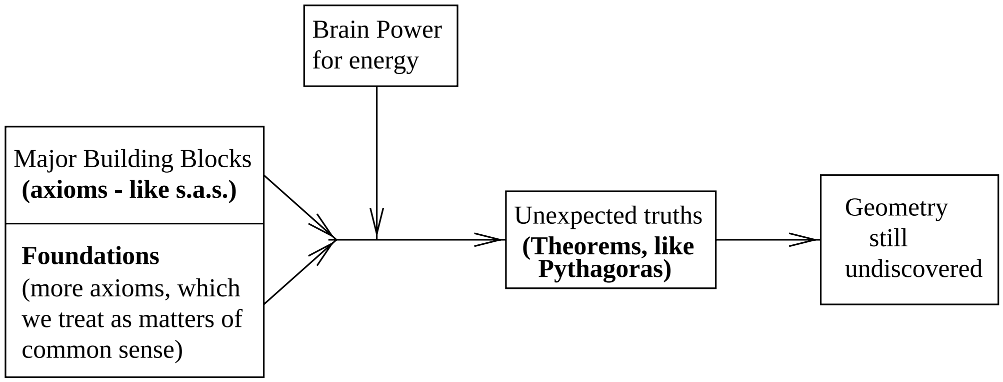


Figure 3: Flowchart for mathematical thinking.

If you do want to read the detailed story, you should consult [12], [14], or [3, chs.1,12,15]. These are challenging mathematical presentations, well worth the effort required for mastery. For now, you can get a good idea of what is involved by reading Section 10.

3.2 Some Simple Geometrical Objects

We shall start with a clean slate; so we don't yet know anything about the angles in a triangle, the area of a circle, etc. In short, we must obey the following law:

Never use any 'substantial' fact which has not yet been stated or proved.[5]

Now let's examine some simple objects and ideas. We may take it as obvious that through any two points A and B (note the capitals) we can draw exactly one straight line, denoted AB or c (lower case for lines). This line can be extended in either direction:

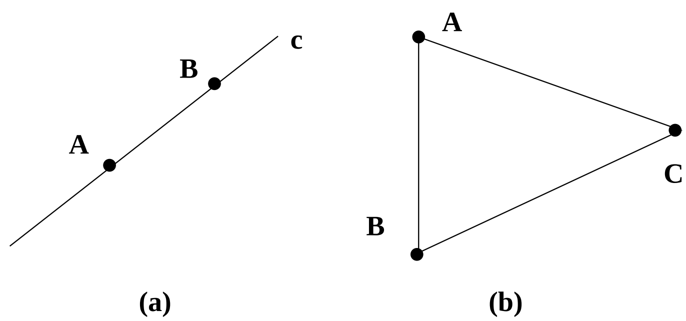


Figure 4: Lines and triangles.

We can also draw a real triangle, denoted $\triangle ABC$, on a sheet of paper (Figure 4 (b)). The thickness of its edges or vertices is often unimportant; for instance, if the triangle represents a metallic plate in some machine, then the lengths AB, BC and CA can be tooled to a desired accuracy. Thus, we may safely suppose that a mathematical triangle $\triangle ABC$ lies on an infinitely thin page (a *plane*) and has *edges* (like AB) and *vertices* (like A) with no thickness.

[5] Just what is 'substantial' is a matter of judgement. There is no avoiding this - only with some experience will you be able to judge what can be assumed and what cannot.

An angle, such as $\angle ABC$, is the figure formed by two segments (or perhaps *rays*) BA and BC emanating from a point B, called the *vertex*:

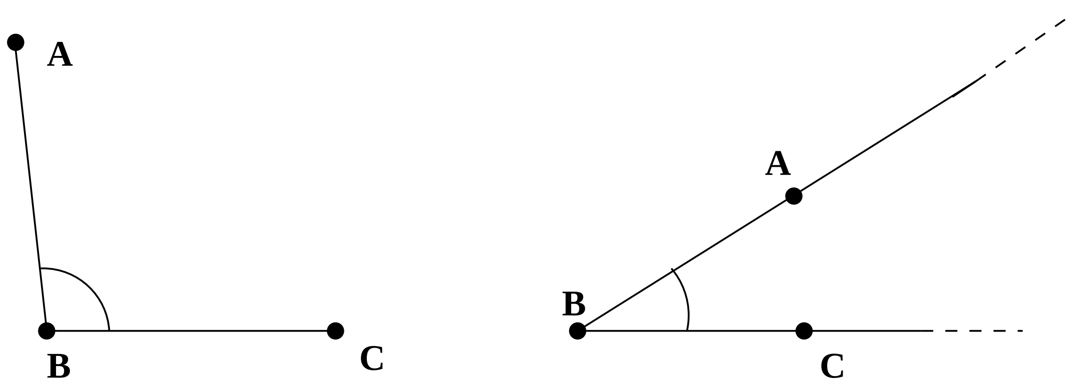


Figure 5: Angles and rays.

A *straight angle* is formed when A, B and C are consecutive points on a straight line:

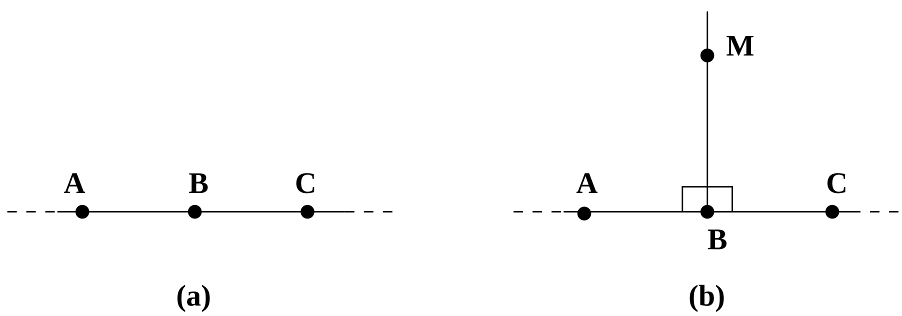


Figure 6: Straight and right angles.

In Figure 6 (b) the ray BM divides (or bisects) the straight angle $\angle ABC$ into two equal angles: $\angle ABM = \angle MBC$. We define a *right angle* as either one of these two equal angles and further say that the line MB is *perpendicular* to the line AC (written $MB \perp AC$).

To measure lengths we use a unit of measurement such as inches or centimeters. Likewise, to measure angles we require a unit such as the *degree*, which is one of the 180 equal parts into which a straight angle may be divided. Thus every straight angle has 180°, and we simply write $\angle ABC = 180°$ in Figure 6. Consequently, every right angle has 90°. This definition of 'degree' is merely a convenience; after all we could define a degree as 1/50th of a straight angle, in which case a right angle would have 25°. Notice that we must not yet use *radian* measure, since it involves π and therefore the circle, about which we still know nothing! (See problem 9c on page 195.)

3.3 Our First Theorem and Proof

We now prove our first simple theorem using these basic ideas. We shall refer to theorems by convenient abbreviations, such as v.o.a.

Theorem 3.1 (v.o.a) *Vertically opposite angles are equal:* $\angle ABC = \angle EBD$ *below.*

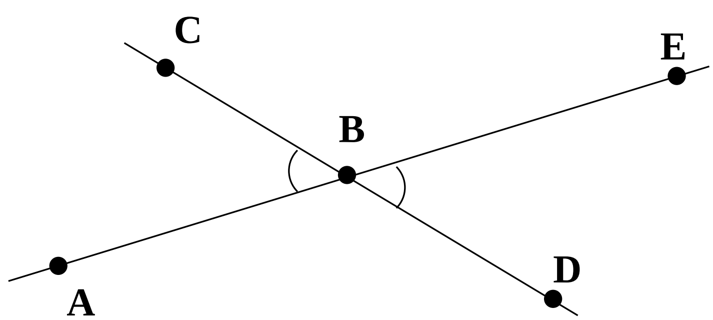


Figure 7: Vertically opposite angles.

Proof: Since $\angle ABE$ is a straight angle,

$$\angle ABE = 180° = \angle ABC + \angle CBE.$$

Likewise $\angle CBD = 180° = \angle EBD + \angle CBE$. Thus by subtracting these quantities we conclude $\angle ABC = \angle EBD$. //

Some Remarks:

(a) Note that // indicates the end of the proof. The aim of a proof is to convince in a logical and clear way. It may be helpful, as in some high school courses, for you to put down a long list of steps with a separate column of justifications. But this is not at all required.

(b) Draw two line segments crossing at the point B, which is the midpoint of each segment. Let's call this figure a *cross.* Now put your pencil tip at B and turn your page through half a turn: note that the cross is unchanged in appearance. The upshot of Theorem 3.1 is that a cross is *symmetrical under a* 180° *rotation.*

3.4 Some Start-up Exercises

Most problems require some sort of proof, which can and should be brief and precise – the main goal is to clearly express a convincing argument. Clear and neat diagrams are very helpful.

For the preliminary exercises below, use only the simple ideas covered up to and including this section in the Notes. Sometimes in these Notes, you may need to refer to a problem solved earlier.

1. If two straight lines intersect, the bisector of any one of the angles, when produced bisects the vertically opposite angle.

2. If two straight lines intersect, the bisectors of two vertically opposite angles form one straight line.

3. AEB and CED are two intersecting straight lines. Prove that the bisector of $\angle AED$ is perpendicular to the bisector of $\angle DEB$.

4. OP, OQ, OR, OS are rays in cyclic order about a common vertex O. Suppose $\angle POQ = \angle ROS$ and $\angle ROQ = \angle SOP$. Prove that PO, OR and also QO, OS are in the same straight line.

3.5 The First Big Axiom

Next we must find some simple, believable facts about our geometrical objects. One of these facts, which seems very reasonable when you look at two identical cardboard triangles, will stand as our first major axiom:

BIG AXIOM I - Side-Angle-Side (s.a.s.). If two triangles $\triangle ABC$ and $\triangle DEF$ have equal corresponding *sides* $AB = DE$, included *angles* $\angle B = \angle E$, and *sides* $BC = EF$, then (we conclude) $AC = DF$, $\angle A = \angle D$ and $\angle C = \angle F$.

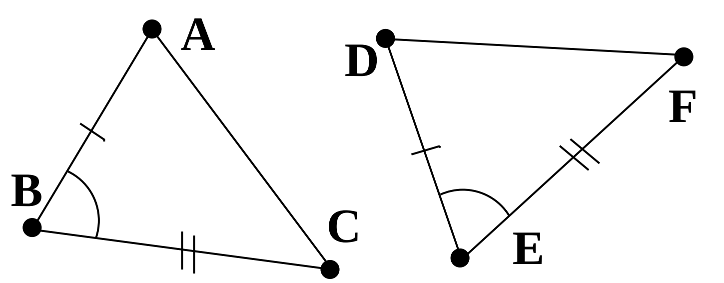


Figure 8: Quantities assumed to be equal are identically marked.

Note that we do not prove this or any other axiom, but rather accept it as a reasonable fact in light of our geometrical experience. Many other geometrical results are logical consequences of the s.a.s. axiom.

When two triangles such as $\triangle ABC$ and $\triangle DEF$ are identical in all respects they are called *congruent* (written $\triangle ABC \equiv \triangle DEF$). It's important to list the vertices in corresponding order: A and D first, B and E second, C and F third. By comparing corresponding parts, one correctly concludes that $AB = DE$, $BC = EF$, $AC = DF$, $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.

4 Building with S.A.S.

The results in this section are proved using only (s.a.s.) and a bit of common sense (the stuff in the *Foundations*).

4.1 The Bridge of Asses (Pons Asinorum)

This obscure title is the classical name for a familiar theorem concerning the isosceles triangle (which has two equal sides).

Theorem 4.1 (P.A.) *If $AB = AC$ in $\triangle ABC$ then $\angle B = \angle C$.*

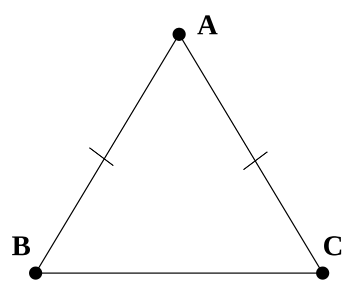


Figure 9: An isosceles triangle.

Proof. Draw line m bisecting $\angle A$ and crossing BC at D (see Figure 10 - equal angles are marked by *'s). In $\triangle BAD$, $\triangle CAD$ we are given $BA = CA$, we construct $\angle BAD = \angle CAD$, and obviously we have $AD = AD$. Hence by (s.a.s.), $\triangle BAD \equiv \triangle CAD$, so $\angle B = \angle C$. //

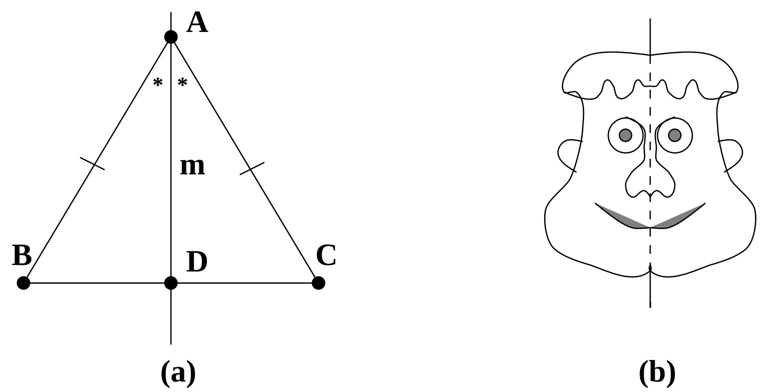


Figure 10: Bilateral symmetry.

4.2 Reflections and Bilateral Symmetry

One way to rephrase P.A. is to say that an isosceles triangle has bilateral symmetry, i.e. equal left and right sides, like most animals (Figure 10(b)).

It is also possible to consider the line m as an infinitely thin mirror, silvered on both sides, which reflects B into C and C into B. In Theorem 4.1 we proved $\triangle BAD \equiv \triangle CAD$ so that $BD = CD$ and $\angle BDA = \angle CDA = 90°$. This suggests the following:

Definition 1 *To* reflect *any point P in a line m we draw through P the line perpendicular to m and on this line choose P' an equal distance from m on the side opposite P. (Thus $\angle PAD = \angle P'AD$ and $PA = P'A$ in Figure 11.)*

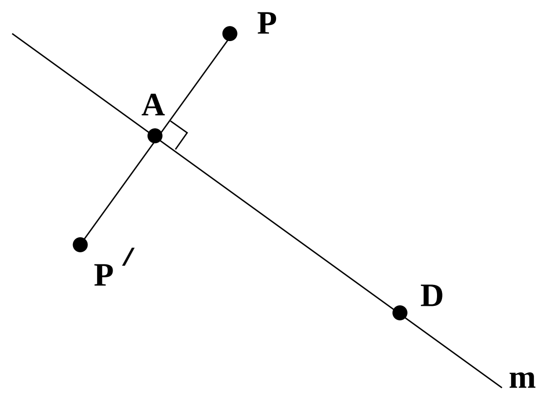


Figure 11: Mirror images.

We say that P' is the *mirror image* (reflected image) of P in the mirror m. Notice that

(a) If P' is the mirror image of P, then P is the mirror image of P'.

(b) Any point like D lying on m is its own mirror image, i.e. $D = D'$.

We now reflect a second point Q, where QQ' meets m at D. It appears (and we shall prove) that the segments PQ and $P'Q'$ are equal in length.[6]

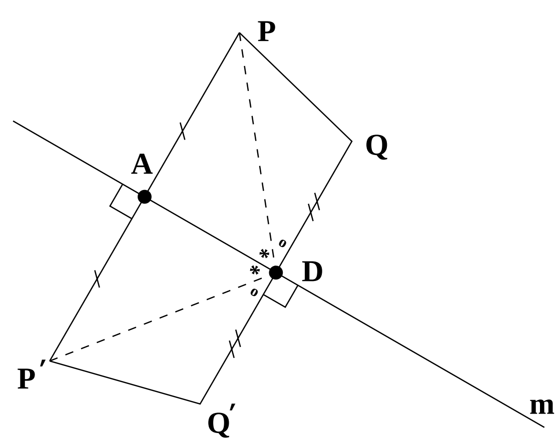


Figure 12: Reflected segments.

[6]Depending on context, PQ could refer to either the *line* through P and Q or to the length of the segment from P to Q.

Theorem 4.2 *Reflections preserve lengths: If P and Q have mirror images P' and Q' by reflection in m, then $PQ = P'Q'$.*

Proof (Figure 12). By definition of reflection, $PA = P'A$ and $\angle PAD = 90° = \angle P'AD$; and of course $AD = AD$. Thus by (s.a.s.) $\triangle PAD \equiv \triangle P'AD$, so $* = *$ and $PD = P'D$. Since $\angle QDA = 90° = \angle Q'DA$ (definition of reflection), we conclude $\circ = \circ$. Thus $\triangle PDQ \equiv \triangle P'DQ'$ by (s.a.s.), so $PQ = P'Q'$. //

4.3 The Triangle Inequality

The supplement of an angle in a triangle is called an *exterior angle*.

Theorem 4.3 (Ext. $\angle$) *In $\triangle ABC$, the exterior angle is larger than either interior opposite angle:*

$$\angle ACD > \angle A \text{ or } \angle B.$$

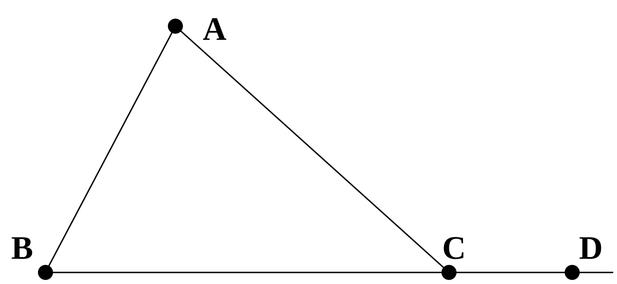


Figure 13: The exterior angle theorem.

Proof . Let M be the midpoint of AC; draw $BM = MP$ and connect PC.

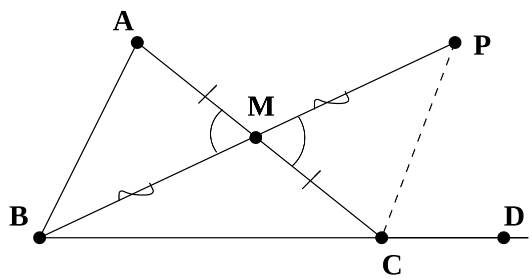


Figure 14: Proof of the exterior angle theorem.

Thus in $\triangle AMB$, $\triangle CMP$ we have $AM = CM$, $\angle AMB = \angle CMP$ (v.o.a.), and $BM = MP$. Hence $\triangle AMB \equiv \triangle CMP$ by (s.a.s.) and $\angle A = \angle MCP$, which is clearly less than $\angle ACD$. The remaining inequality is left as problem 2 on page 38. //

We next *prove* a fundamental property of space, namely that the shortest distance between two points is measured along the straight line joining them.

Theorem 4.4 ($\triangle$ inequality) *In* $\triangle ABC$, $AC < AB + BC$.

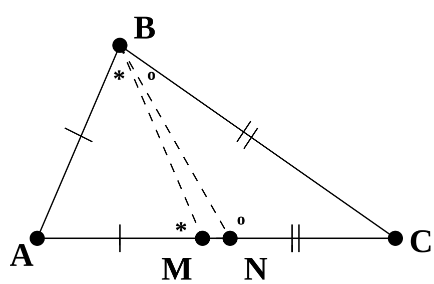


Figure 15: A proof of the triangle inequality.

Remark on the Proof. We shall prove this by *contradiction.* Either our theorem is true or it isn't, so we tentatively assume the theorem to be false. Next we try to logically deduce an obviously false statement (a contradiction). If we do succeed in getting a contradiction, and if the universe isn't playing tricks on us, then we are forced to deny our tentative assumption that the theorem is false. In short, it must be true!!

Proof. Suppose on the contrary that $AC \geq AB + BC$. Then we can mark off $AB = AM$ and $NC = BC$ on the segment AC. Connect MB and NB. By (P.A.), $* = *$ and $\circ = \circ$. Applying (Ext. $\angle$) to $\triangle BMC$ and $\triangle BNA$, we find that

$$* > \angle MBC \geq \circ \text{ and } \circ > \angle NBA \geq *.$$

Hence, $* > *$, which is impossible. Thus by contradiction, $AC < AB + BC$. //

4.4 A Neat Application

The geometrical consequences of (s.a.s.) seem so far to be pretty dull stuff. Here is a quite unexpected application.

Problem: Two towns P and Q on the same side of a straight river m require a water plant W. Where should the plant be located so that the total length of the water pipes is as small as possible?

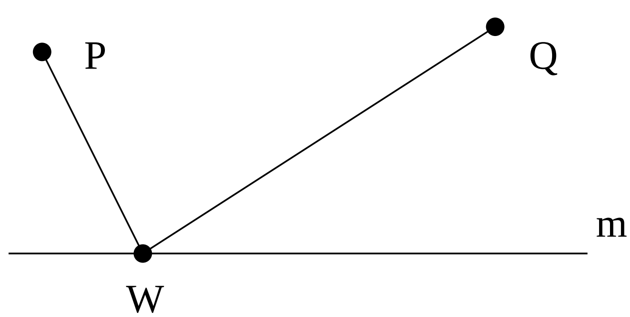


Figure 16: Locating the water plant.

Solution: We must minimize $PW + WQ$. Reflect Q in m. By Theorem 4.2, $WQ = WQ'$. The total pipe length is thus $PW+WQ = PW+WQ'$. However, by applying (△ inequality) to $\triangle PWQ'$ we find that $PW + WQ' > PQ'$. Hence the smallest that the pipe length could be is PQ', and to get this we place the water plant at the point M where PQ' crosses m. Done! //

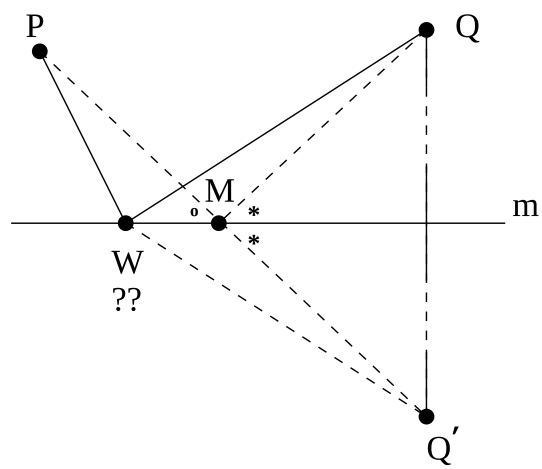


Figure 17: Minimizing the total path length.

The same geometrical idea can be used in a quite different application. *Fermat's principle* in optics states that a light ray will follow the path of shortest time required. Now suppose in Figure 17 that m is a mirror which reflects a light ray from P so that it eventually passes through Q. By Fermat's principle the ray must strike the mirror at M. However, we know from the proof of Theorem 4.2 that $* = *$, whereas $* = \circ$ by (v.o.a.). Thus $\circ = *$ and we have proved the

Law of Reflection. When a light ray is reflected by a straight mirror m, the angle of incidence equals the angle of reflection.

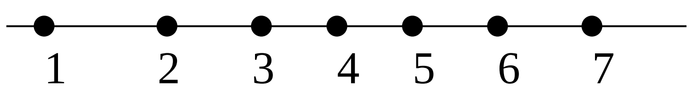
Remark. A billiard ball with no spin will bounce in the same way off the banks of a billiard table.

4.5 Sylvester's Theorem

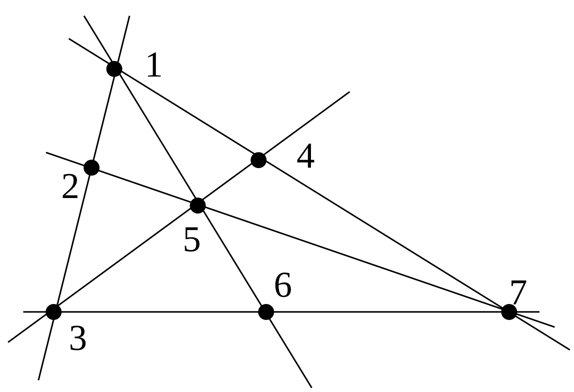
In 1893, the English mathematician J. J. Sylvester posed the following problem, which was not solved until about 1933 by T. Gallai. In 1948, L. M. Kelly gave the elegant proof outlined below.

Before reading further, attempt the following exercise to get a feeling for the situation: try to draw say $n = 7$ points in the plane such that the line containing each *pair* of the n points contains at least one other of the n points.

After a while, you may think of the following solution, which isn't too exciting:



Here is another, almost correct, attempt:

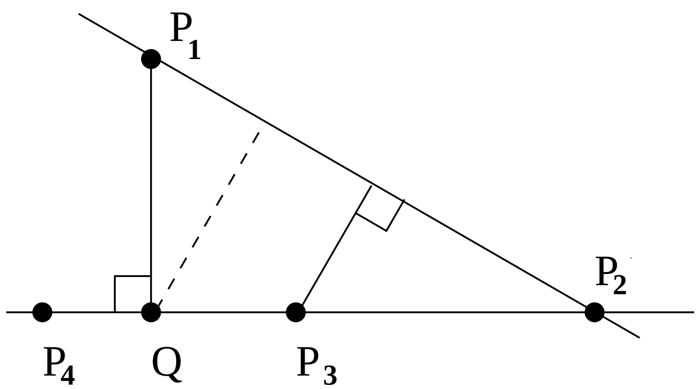


What's wrong?

In fact, Sylvester's Problem asserts that we can never draw the points as required.

Sylvester's Problem: Given any $n \geq 3$ points in the plane, not all on a single line, show that there is at least one line containing exactly two of the points.

Proof. There are only *finitely* many points P_1, ..., P_n and joining lines P_1P_2, P_1P_3, ..., $P_{n-1}P_n$. Thus, there must be a point, say P_1 and line, say P_2P_3 (not containing P_1) for which the distance P_1Q from point to line is the *smallest* such distance which occurs. We claim that line P_2P_3 is the desired 'special' line. Indeed, suppose for the moment that line P_2P_3 contains another of the given points, say P_4:



Then *two* of the points, say P_2 and P_3, must lie on the same side of Q. But clearly the distance from P_3 to line P_1P_2 (see the drawing) must be smaller than the distance from Q to line P_1P_2, which in turn is smaller than the distance P_1Q from P_1 to line P_2P_3.[7] This contradicts our choice of the minimum such distance. Line P_2P_3 must indeed be a special line. //

Recent History. The problem continues to be investigated to this day. However, we mention only that in 1958, Kelly and Moser proved that there must be at least $3n/7$ 'special' lines.

[7]These rather subtle inequalities can in fact be proved as a consequence of the s.a.s axiom.

4.6 Other Congruence Theorems

Many problems with triangles require different congruence theorems. The following standard results follow from (s.a.s.).

Theorem 4.5 (a.s.a.) *If $\triangle ABC$ and $\triangle DEF$ have corresponding equal angles $\angle A = \angle D$, sides $AB = DE$ and angles $\angle B = \angle E$, then $\triangle ABC \equiv \triangle DEF$ (note that the equal sides connect the equal pairs of angles).*

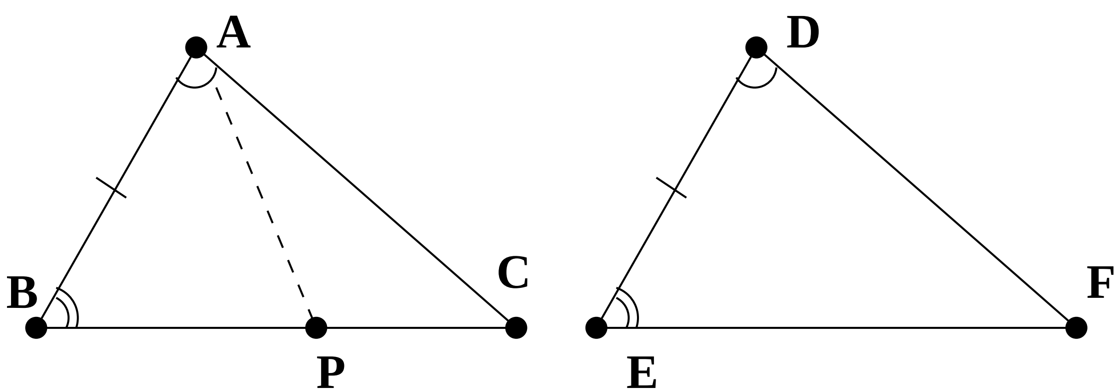


Figure 18: Congruence with a.s.a.

Proof. On line BC draw $PB = FE$. (See Figure 18, in which P has been deliberately misplaced.) Since we are given $\angle B = \angle E$ and $BA = ED$ we conclude by (s.a.s.) that $\triangle PBA \equiv \triangle FED$. Thus $\angle BAP = \angle EDF = \angle BAC$. Hence $P = C$ and $\triangle CBA \equiv \triangle FED$. //

Theorem 4.6 (s.s.s.) *If $\triangle ABC$ and $\triangle DEF$ have equal corresponding sides $AB = DE$, $AC = DF$, $BC = EF$, then $\triangle ABC \equiv \triangle DEF$.*

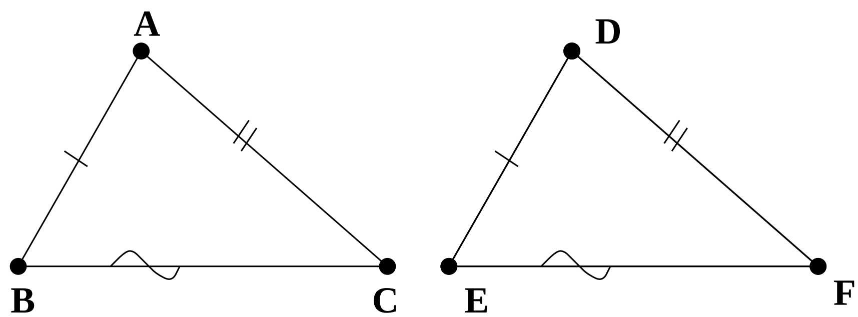
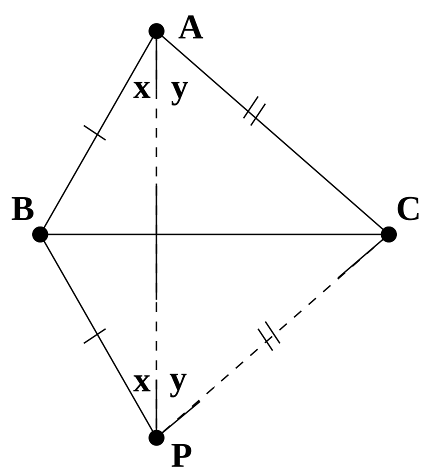


Figure 19: Congruence with s.s.s.

Proof. Draw $\angle CBP = \angle FED$ with $BP = ED = AB$. By (s.a.s.), $\triangle CBP \equiv \triangle FED$, so $PC = DF = AC$:



Connect AP and deduce that $x = x$, $y = y$ (by P.A.). Thus $\angle A = x + y = \angle P = \angle D$, so $\triangle ABC \equiv \triangle DEF$ by (s.a.s.). //

Remark. Theorem 4.6 demonstrates that a triangle is **rigid** - for if it is made from metal rods, then its angles cannot be distorted.

Theorem 4.7 (R.h.s.) *If right triangles $\triangle ABC$ and $\triangle DEF$ have equal hypotenuses $AC = DF$ and sides $BC = EF$, then $\triangle ABC \equiv \triangle DEF$.*

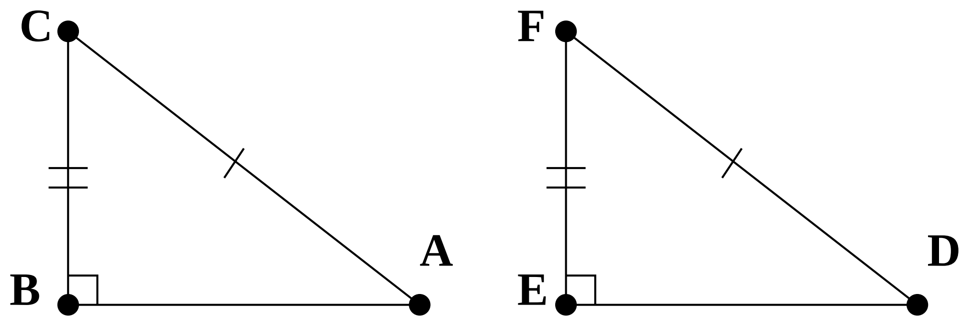
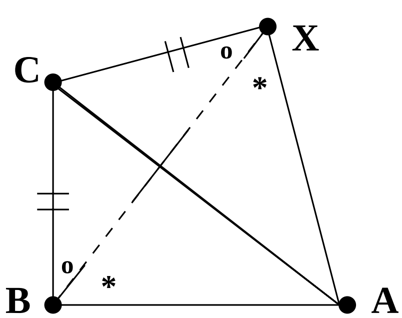


Figure 20: Congruence with right triangles.

Proof. Construct AX so that $\angle CAX = \angle FDE$ and $AX = DE$. Thus $\triangle CAX \equiv \triangle FDE$ by (s.a.s.), so $CX = FE = CB$ and $\angle X = 90^\circ = \angle B$.



Thus, $\circ = \circ$ (by P.A.) and hence $* = *$. Thus $BA = XA = ED$. (We use the converse to P.A., which is easily proved). Finally, $\triangle ABC \equiv \triangle DEF$ by (s.a.s.), since we now know $AB = DE$. //

4.7 Our first look at parallel lines.

Definition 2 *Two lines* b *and* c *are* parallel *(written* $b \| c$*) if they lie in the same plane but do not intersect, or if they are the same line. A third line* m *is called a* transversal *if it intersects both* b *and* c*:*

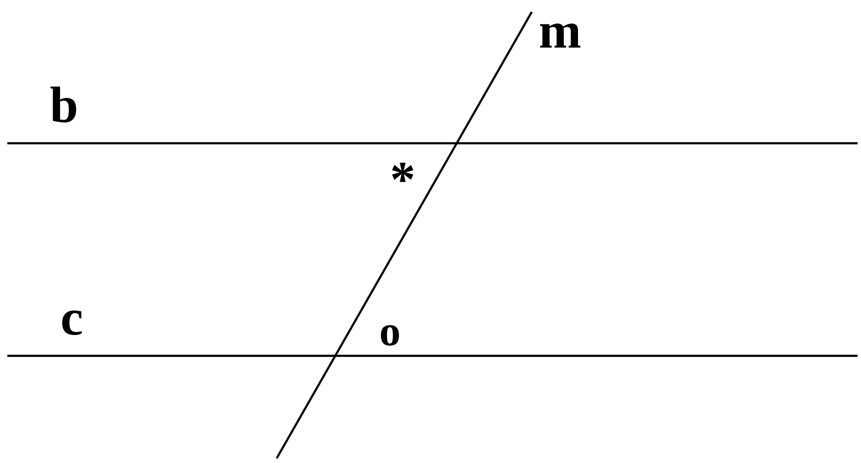


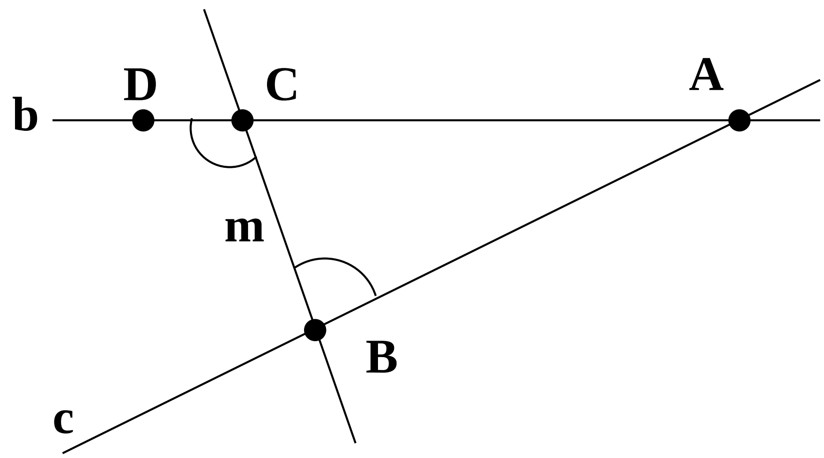
Figure 21: Parallel lines and a transversal.

The angles marked * and $\circ$ are called *alternate* angles.

Notice that although we have defined 'parallel lines', we *still do not know that they actually exist*! After all, we can define anything we want, but just doing so is no guarantee that our definition is useful or makes any sense. Let's investigate further.

Theorem 4.8 (equ. alt. $\Rightarrow$ par.) *If a transversal m makes equal alternate angles with lines b and c, then b is parallel to c.*

Proof (by contradiction). Suppose b and c intersect at A:



We are given $\angle DCB = \angle CBA$. However, by (Ext. $\angle$), $\angle DCB > \angle CBA$. Because of this contradiction we conclude that lines b and c don't intersect. //

Corollary 4.9 *If C is a point not on line c, then there exists a line b through C which is parallel to c.*

Proof.

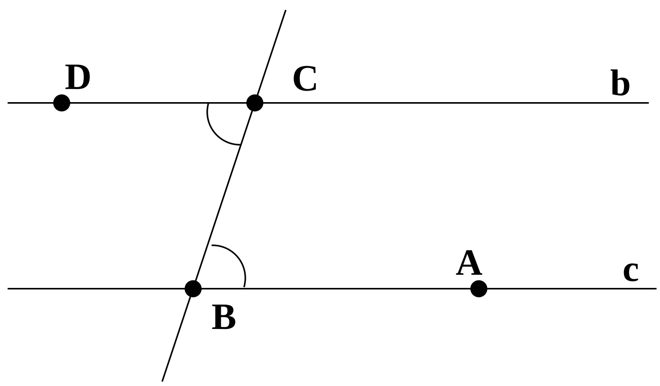


Figure 22: Parallel lines exist!

Join C to any point B on $c = BA$ and construct $\angle BCD = \angle CBA$ as shown. It follows that $b = DC$ is parallel to c. //

We have thus proved that non-intersecting, i.e. parallel, lines really do exist, in abundance!

4.8 Absolute Geometry.

Using only the most obvious axioms, such as (s.a.s), as a starting point, we have nevertheless accomplished quite a lot. The body of geometrical theorems which depend only on these basic axioms, and which *do not* involve any more specific discussion of the nature of parallelism, is sometimes called **absolute geometry**.

In a sense, absolute geometry is a bit like **Euclidean geometry** without frills. But even that description is a little inaccurate. In order to get Euclidean geometry, we are forced to accept a totally new axiom which governs in a more precise way the behaviour of parallel lines.

The astonishing insight of 19th–century mathematics was that we can equally well accept a very different parallelism axiom and still have a perfectly logical, but rather different, **non-Euclidean geometry**. We will briefly return to this issue in section §6 below.

For now, here are a few more introductory, 'warm-up' exercises, followed by several large collections of problems in absolute geometry. Try as many as you can.

4.9 On doing problems and constructions in geometry.

Instructions.

- Most problems require some sort of proof.

  1. Be brief and precise. This does not always mean that your answer must be written in a very formal way. The main goal is to clearly express a convincing argument.
  2. Submit neat, clear diagrams.
  3. For problems in absolute geometry, you must be very careful to use the results described in the text, up to §4.8.

     Otherwise, in later problem sets, feel free to use any pertinent result from class or another source (with reference). Most exercises will require only simple results, occurring early in your notes.

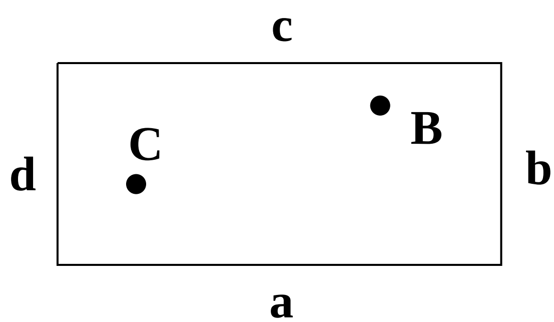
- Some special instructions for ruler and compasses (R-C) constructions.

  1. You may use only compasses and a straight edge (i.e. one side of a ruler, ignoring its cm. or inch marks). The legitimate usage of these instruments is described in Section 24.
  2. Many such problems require a proof that your construction does what is claimed.

4.10 Some Introductory Problems

These exercises will help you absorb the geometric ideas introduced in the text to this point. You need not be too concerned with axiomatics for the following three questions.

1. (a) In the rectangular billiard table below you must bounce the cue ball C off bank a, then bank b so as to hit the black ball B. Copy the figure and explain how to do this:



   (b) Where would you aim if you had to hit bank a, bank b, bank c, then ball B? (Explain briefly.)

   (c) Where would you aim if you had to hit bank a, bank b, bank c, bank d, then return to the original position of the cue ball C? (Explain briefly.)

2. Draw the x and $y-$axes as accurately as you can If you want, use lightly ruled graph paper. Let m_1 denote the $x-$axis and m_2 the $y-$axis.

   (a) What is the reflected image of the point $[2,3]$ in line m_1? in line m_2 ?

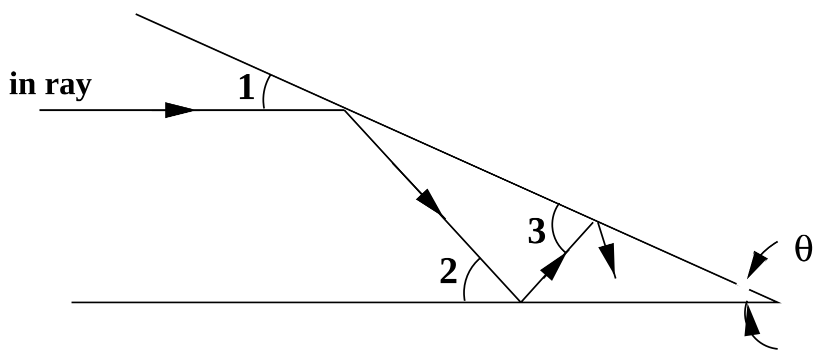
   (b) Answer part (a) for any point $[x,y]$.

(c) We could reflect a point in m_1, then reflect the result in m_2, etc., thereby moving a point all over the plane. Describe what happens if we reflect the point $[3, 4]$ in m_2, then m_1, then m_2, then m_1, then m_2, etc.

(d) In part (c), what figure does the moving point trace out?

(e) Instead of starting with $[3, 4]$, with what point or points could you start to make each side of the resulting figure equal?

3. Two mirrors are inclined at angle θ and a light ray enters parallel to one mirror:



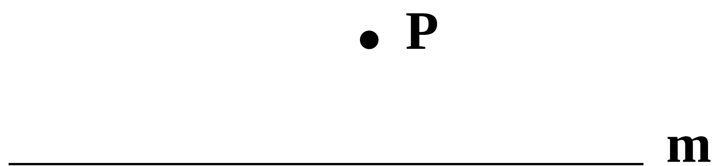
(a) Determine angles 1, 2, 3, 4, etc. What is the n^{th} such angle?

(b) Sketch the complete path of the light ray when $\theta = 15°$ and when $\theta = 36°$.

4.11 Questions in Absolute Geometry.

In each of the following problems you may use only results covered in the Notes up to, and including, § 4.8. (In other words, the angle sum in a triangle, similarity, trigonometry, Pythagoras are **not** needed and are **not** allowed.)

1. (a) Given a line m and a point P, describe how to construct the mirror image P' of P in m using *only* compasses twice.



   (b) Prove that your construction fulfills the definition of reflection, using only (s.a.s.) and (P.A.). (In short, prove that your method actually works! If you are unsure how compasses can be legally used, read Section 24.2.)

2. Review Theorem 4.3 (Ext. $\angle$) of the notes, where we proved $\angle ACD > \angle A$. Using only this and previous theorems, prove also that $\angle ACD > \angle B$.

3. Any point on the right bisector of a straight line segment is equidistant from the ends of the segment.

4. State and prove the converse of the proposition in the previous exercise.

5. The point of intersection of the right bisectors of two sides of a triangle is equidistant from the three vertices.

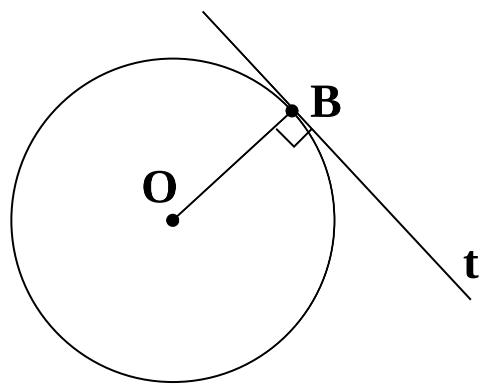
6. The right bisectors of the three sides of a triangle pass through one point.

7. $KLMN$ is a quadrilateral in which $KL = MN$ and $\angle L = \angle M$. Prove that $\angle K = \angle N$.

8. If two circles intersect, the straight line which joins their centres is the right bisector of their common chord.

9. If M is the midpoint of a chord AB of a circle with centre O, then $OM \perp AB$.

10. Given: a circle of centre O and radius OB, draw the line t through B, perpendicular to OB:



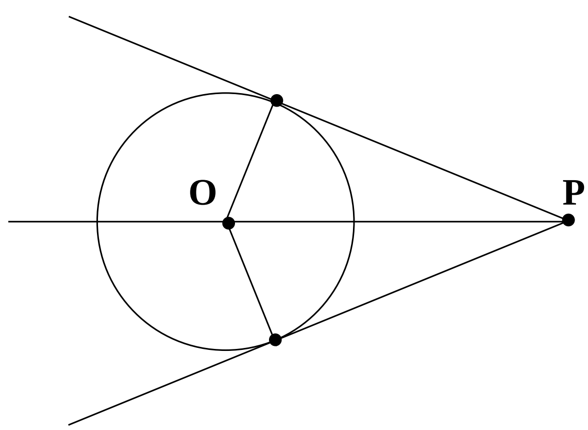
Prove that the line t intersects the circle at just one point, namely B. (Hence t is called a **tangent line**.)

(Hint—prove this by contradiction; theorem 4.3 will be then be useful.)

11. If two tangents are drawn to a circle from an external point P then

    (a) the tangents are equal.

    (b) the line joining P to the centre bisects the angle between the tangents.



12. The diagonals of a rhombus bisect each other at right angles.

13. Suppose AB, CD, EF are three diameters of a circle. Prove that $\triangle ACE \equiv \triangle BDF$.

14. AB and CD are two equal straight lines. The right bisectors of AC and BD meet at E. Prove that $\triangle EAB \equiv \triangle ECD$.

15. No two straight lines drawn from the vertices of a triangle and terminated in the opposite sides can bisect each other.

16. Any point which is equidistant from the arms of an angle, lies on the bisector of that angle.

17. The bisectors of the angles of a triangle are concurrent, i.e., they all pass through one point.

4.12 More Questions in Absolute Geometry.

In each of the following problems you may use only results covered in the Notes up to, and including, § 4.8. (In other words, the angle sum in a triangle, similarity, trigonometry, Pythagoras are **not** needed and are **not** allowed.)

You may also refer to other problems from these problem sets on absolute geometry.

1. **Theorem.** Suppose $AB > AC$ in $\triangle ABC$. Then $\angle C > \angle B$, i.e., in a triangle, a larger side is opposite a larger angle.

   (Hint: On AB cut off $AD = AC$, which *is* possible since $AB > AC$.)

2. **Theorem.** (Converse to previous Theorem.) Suppose $\angle C > \angle B$ in $\triangle ABC$. Then $AB > AC$, i.e., the larger angle lies opposite the larger side.

   (Hint: Try contradiction plus previous theorem.)

3. **Theorem.** Suppose AP is perpendicular to line BCP, with C between B and P. Then $AB > AC > AP$ (i.e. the shortest distance from a point A to line BC is along the perpendicular, and other distances increase as expected).

4. **Theorem.** Suppose $AB = DE$ and $BC = EF$ for $\triangle ABC$ and $\triangle DEF$, but $\angle B > \angle E$. Then $AC > DF$.

5. **Theorem.** Suppose two chords AB and CD in a circle are equidistant from the centre O. Then $AB = CD$.

6. **Theorem.** Suppose $AB = CD$ are chords in a circle with centre O. Then AB and CD are equidistant from O.

7. The sum of the diagonals of any quadrilateral is greater than the sum of either pair of opposite sides.

8. The sum of the sides of a quadrilateral is greater than the sum of its diagonals.

9. If any point within a triangle be joined to the ends of one side, the sum of the lengths of the joining segments is less than the sum of the other two sides of the triangle.

10. If any point within an equilateral triangle be joined to each of the vertices, the sum of the lengths of any two of the joining segments is greater than the third.

11. The sum of any two sides of a triangle is greater than twice the median drawn to the third side.

12. If the bisector of the vertical angle of a triangle bisects the base, prove that the triangle is isosceles.

13. Suppose O is any point on the bisector of the angle $\angle BAC$; a circle with centre O cuts each of the lines AB, AC in two points; the points on these lines nearest to A are E and F. Prove $AE = AF$.

14. Say P is a point equidistant from the arms of an angle $\angle AOB$. Prove that PO bisects $\angle AOB$.

15. Suppose $ABCD$ is a quadrilateral having $AB = CD$, but $\angle BCD$ is greater than $\angle ABC$; prove that BD is greater than AC.

16. In $\triangle ABC$ suppose that $AB > AC$. Equal distances BD and CE are cut off from BA, CA respectively. Prove that $BE > CD$.

17. In the triangle PQR, $PQ > PR$ and T is the middle point of QR. If any point A on the median PT be joined to Q and R, prove $AQ > AR$.

18. $ABCD$ is a quadrilateral having $AB = AD$ and $\angle B = \angle D$. Prove that AC bisects BD at right angles.

19. If two sides AB, AC of a triangle ABC are equal and BD, CE are drawn perpendicular to AC and AB, and intersect in O, prove that AO bisects the angle A.

20. A straight line is drawn to cut the outer of two concentric circles at A and B and the inner at X, Y. Prove $AX = BY$.

21. AB and AC are two equal chords of a circle. Prove that the bisector of $\angle BAC$ passes through the centre of the circle.

22. A straight line cannot cut a circle at more than two points. (Hint: use the indirect method, i.e. proof by contradiction.)

23. A circle is known to pass through a point P and have its centre on a given line AB. Find another point which must be on the circumference.

24. If two chords of a circle intersect each other, and make equal angles with the diameter drawn through their point of intersection, they are equal.

25. Tangents to a circle at the ends of a diameter are parallel.

5 Introducing the Euclidean Axiom of Parallelism.

In order to better understand just how parallel lines work in Euclidean geometry, you should re-read § 4.7. There we were able to prove the key Theorem 4.8:

(equ. alt. ⇒ par.) If a transversal m makes equal alternate angles with lines b and c, then b is parallel to c.

This (absolute!) theorem is in no way based on any explicit axiom governing the behaviour of parallel lines. We also verified the following Corollary 4.9:

If C is a point not on line c, then there exists a line b through C which is parallel to c.

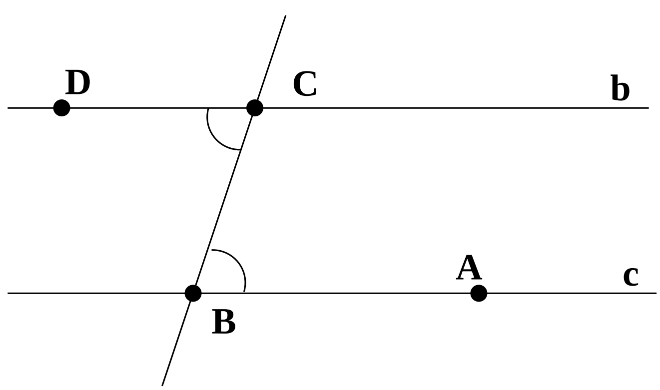


Figure 23: Through any point C there does pass a line parallel to a given line c.

Considering Figure 23, we intuitively believe that DC is the *only* line through C parallel to c. (See Figure 30 a few pages below for another possibility.) However, we *cannot* prove that our intuition is correct using just (s.a.s.) and common sense. Instead, we are forced to introduce a new axiom.

BIG AXIOM II - The Parallelism Axiom (Par. Ax.). If C is any point and c any line, then there passes through C *exactly one* line b parallel to c.

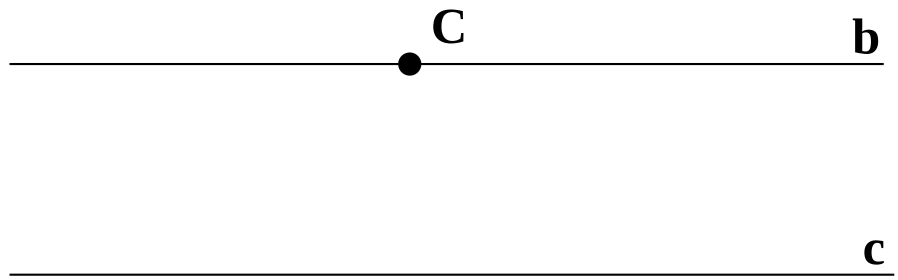


Figure 24: The parallelism axiom.

Thus the axiom asserts that any *other* line through C will meet c in a single point, instead of being parallel to c. We could restate the axiom using the phrase '*at most one* line b parallel to c' because Corollary 4.9 already gives us one such parallel line. We shall soon find several important and well known consequences of this axiom, beginning with another 'obvious result' which nonetheless requires a proof:

Theorem 5.1 *If $b \| c$ and $c \| a$, then $b \| a$.*

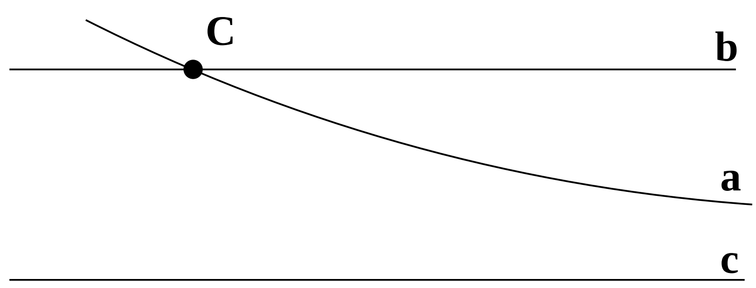


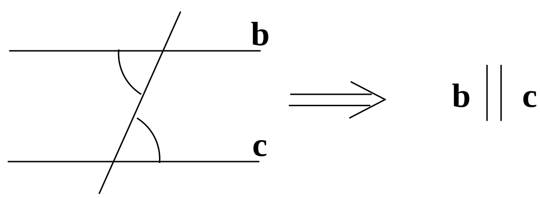
Figure 25: A 'strange' picture used in the proof: draw the 'normal' version yourself.

Proof (by contradiction). If $b = a$ there is nothing to prove. So assume b and a intersect at one point C. But then b and a are two different lines through C, even though both are parallel to line c by assumption. This contradicts (Par. Ax.). Hence b and a do not intersect and are therefore parallel. //

Next we recall that Theorem 4.8 asserts that

'**If** alternate angles are equal, **then** lines b and c are parallel'.

or



This does not necessarily imply the *converse* statement, in which we **assume** $b \| c$ and **conclude** the equality of alternate angles. Fortunately, for the sake of Euclidean geometry, this converse statement is true, although to prove it we must use the parallelism axiom.

Theorem 5.2 *(converse to Theorem 4.8). If parallel lines b and c are cut by a transversal m, then the alternate angles are equal: $\angle BCD = \angle CBA$.*

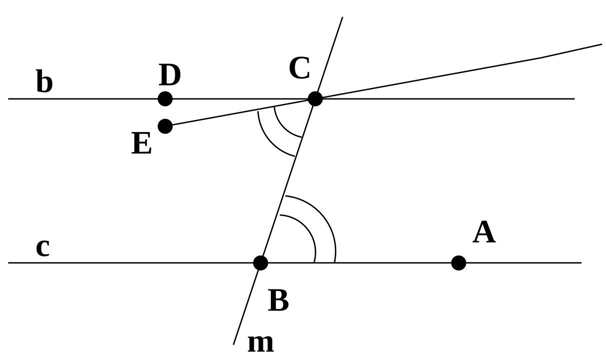


Figure 26: Parallel lines force equal alternate angles.

Proof. Draw CE with $\angle BCE = \angle CBA$. By Theorem 4.8, CE is parallel to c. But the (Par. Ax.) states that there is exactly one line through C parallel to c. Since b is such a line, $b = CE$ (and our picture is incorrect). Hence $\angle BCD = \angle BCE = \angle CBA$. //

Corollary 5.3 *In Figure 27 below,*

(a) $b \| c$ *if and only if* $\angle 1 = \angle 2$

(b) $b \| c$ *if and only if* $\angle 2 = \angle 4$

(c) $b \| c$ *if and only if* $\angle 2 + \angle 3 = 180°$.

Proof. Combine Theorem 4.8 and Theorem 5.2 to prove (a). Parts (b) and (c) follow by (v.o.a.) and the definition of a 180° angle. //

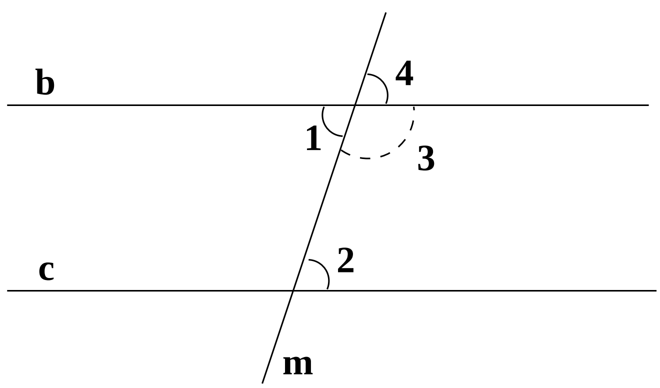


Figure 27: Parallel lines and certain angles.

Corollary 5.4 *In $\triangle ABC$, $\angle A + \angle B + \angle C = 180°$.*

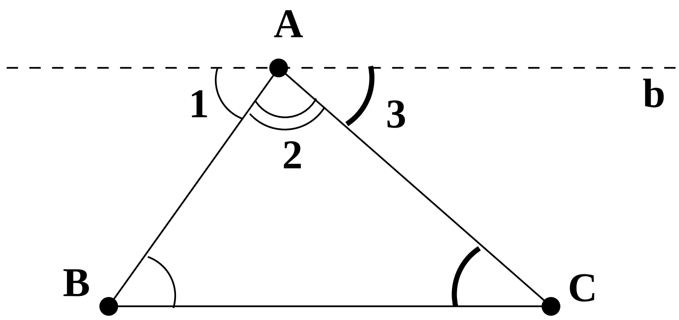


Figure 28: Angle sum in a triangle.

Proof. Let b be the line through A parallel to BC. By Theorem 5.2, $\angle B = \angle 1$ and $\angle C = \angle 3$, so $\angle A + \angle B + \angle C = \angle 1 + \angle 2 + \angle 3 = 180°$. //

This last result is perhaps the best known Theorem in elementary geometry. We are able to prove this theorem, and the theorem is true, only because we have assumed (Par. Ax.).

Corollary 5.5 *(Special Ext. $\angle$). In $\triangle ABC$, an exterior angle equals the sum of the two opposite interior angles: $\angle ACD = \angle A + \angle B$.*

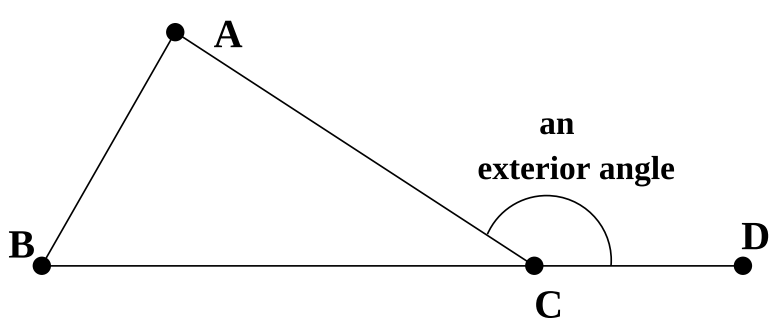


Figure 29: The exterior angle in the Euclidean case.

Proof .

$$\begin{array}{rl} \angle ACD + \angle ACB = 180^\circ & \text{(straight angle)} \\ \angle A + \angle B + \angle ACB = 180^\circ & \text{(Cor. 5.4)} \\ \text{Thus } \angle A + \angle B = \angle ACD. & // \end{array}$$

Note on Cor. 5.5. This result depends ultimately on (s.a.s.) and (Par. Ax.). Theorem 4.3 is a weaker version of this result, but then again Theorem 4.3 depends only on (s.a.s.).

6 A Digression on the Parallelism Axiom.

The Parallelism Axiom rules out the sort of picture suggested in Figure 30, in which there is *more than one* line parallel to c and through C. (Neither b_1 nor b_2 intersects c; it is an inevitable limitation of drawing in the Euclidean plane that these lines seem to 'curve'.)

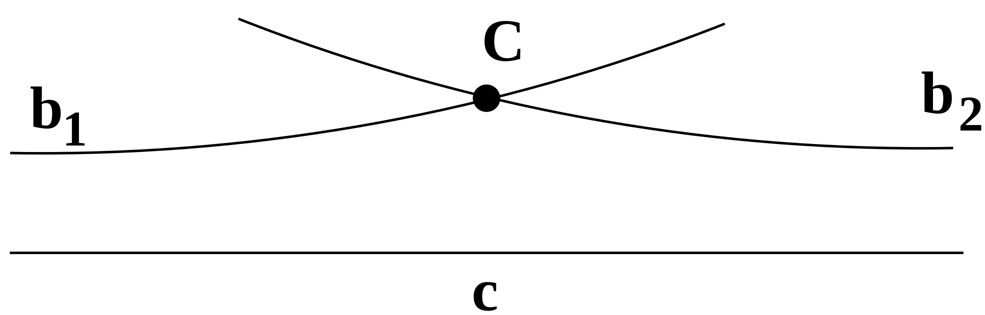


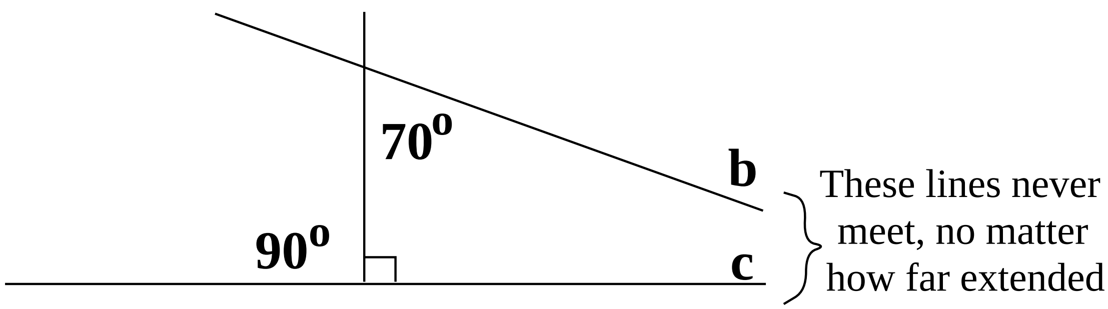
Figure 30: Non-Euclidean parallels.

Though it may seem 'obvious' that Figure 30 cannot occur in nature, we still must not use any fact which has not been established as an axiom or theorem. Hence, we are forced to assume the Parallelism Axiom in our development of Euclidean geometry.

Remarkably, there is, in fact, a wonderful type of geometry called *hyperbolic geometry* in which (s.a.s) is true but the usual Parallelism Axiom is false! Indeed, the situation depicted in Figure 30 can and does occur.

Thus in hyperbolic geometry, Theorems 3.1 to 4.8, which depend only on (s.a.s.) among the crucial axioms, are true. But many other well known theorems from Euclidean geometry are false in hyperbolic geometry. For example, in hyperbolic geometry the sum of the angles of a triangle is always less than 180° and can even be as small as 0° !! In other words, hyperbolic geometry is *non-Euclidean* since many common theorems which are based on the Euclidean Parallelism axiom are no longer true.

As another example, consider the following figure, which does exist in the hyperbolic plane:



Our Euclidean intuition protests, 'no way!'. The point is that our diagram is printed on a Euclidean page, so that we should not expect it to accurately represent the truth in hyperbolic geometry. There is nothing wrong with the mathematics.

So none of this strange mathematics means that hyperbolic geometry is 'incorrect'; it too can be built up from axioms in a logical way (but, of course, a different parallelism axiom is required). Certainly, hyperbolic geometry does not apply to everyday measurements. Amazingly, however, it does arise quite naturally in Einstein's theory of special relativity and in many other branches of mathematics. For more information consult [12, ch.6] or [3, ch. 16].

7 Parallelograms and The Intercepts Theorem

Definition 3 *A* parallelogram *is a quadrilateral whose opposite sides are parallel.*

Theorem 7.1 *The opposite sides of a parallelogram are equal:*

$$AB = CD \text{ and } AD = CB$$

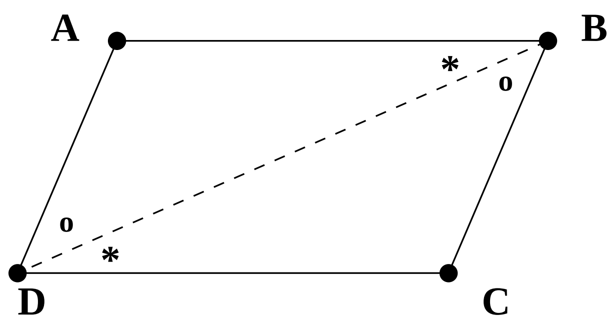
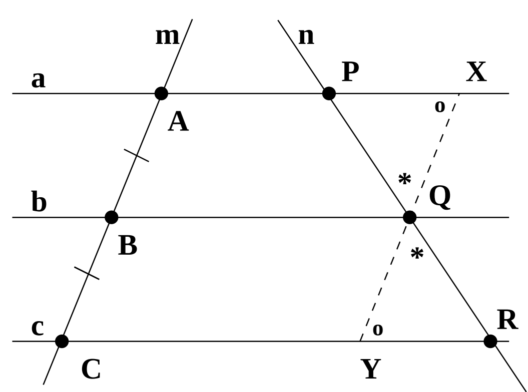


Figure 31: Opposite sides in a parallelogram are equal.

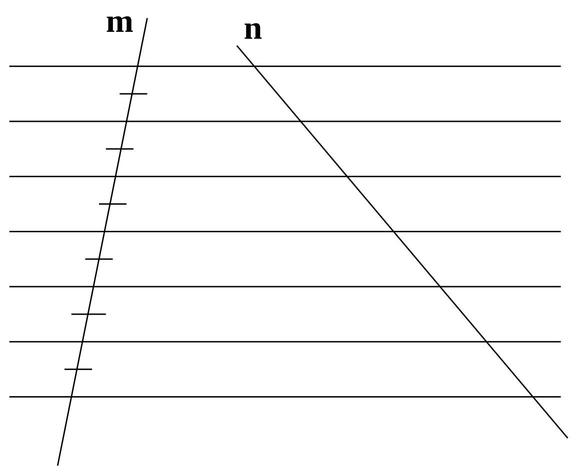
Proof. Connect BD. Since $AB \| DC$ and $AD \| BC$ we conclude from Theorem 5.2 that $* = *$ and $\circ = \circ$. Hence $\triangle ABD \equiv \triangle CDB$ (by a.s.a.), so $AD = CB$ and $AB = CD$. //

Theorem 7.2 *(Intercepts Theorem). If three parallel lines a, b, c make equal intercepts with a transversal m, then the intercepts with any other transversal n are also equal.*



Proof. We are *given* $AB = BC$ and must *prove* $PQ = QR$. Draw line XQY parallel to AC as shown. Since $AXQB$ is a parallelogram we conclude by Theorem 7.1 that $XQ = AB$. Similarly $YQ = BC$, so $XQ = AB = BC = YQ$. By (v.o.a.) $* = *$, and by Theorem 5.2, $\circ = \circ$. Hence $\triangle XQP \equiv \triangle YQR$ by (a.s.a.) and $PQ = RQ$. //

Corollary 7.3 *If any number of parallel lines make equal intercepts with a transversal m, then they make equal intercepts with any other transversal n. (An example with seven parallel lines is illustrated below).*



Proof. By Theorem 7.2, consecutive intercepts on n are equal, so they equal one another. //

Theorem 7.4 *(The Ratio Theorem). If PQ is parallel to BC in $\triangle ABC$, with P on AB and Q on AC, then* $\dfrac{AP}{PB} = \dfrac{AQ}{QC}$

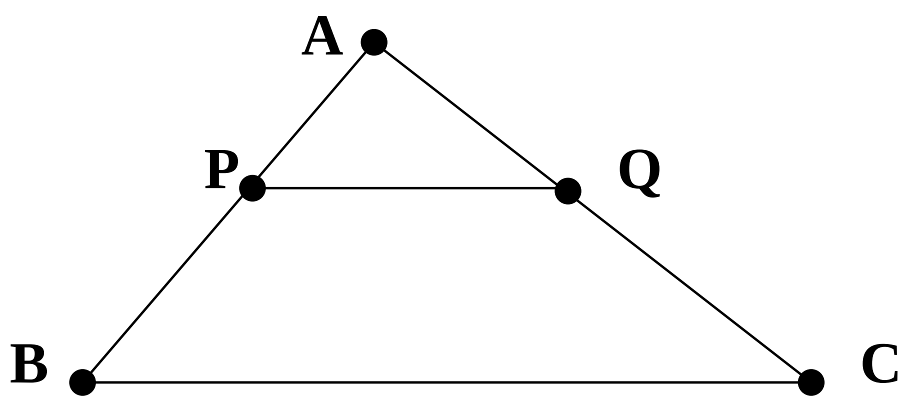


Figure 32: The ratio theorem.

Proof. Suppose that $\dfrac{AP}{PB} = \dfrac{p}{q}$, where p and q are positive integers. (Thus $\dfrac{p}{q}$ is a rational number; the case $\dfrac{p}{q} = \dfrac{3}{4}$ is shown in Figure 33).

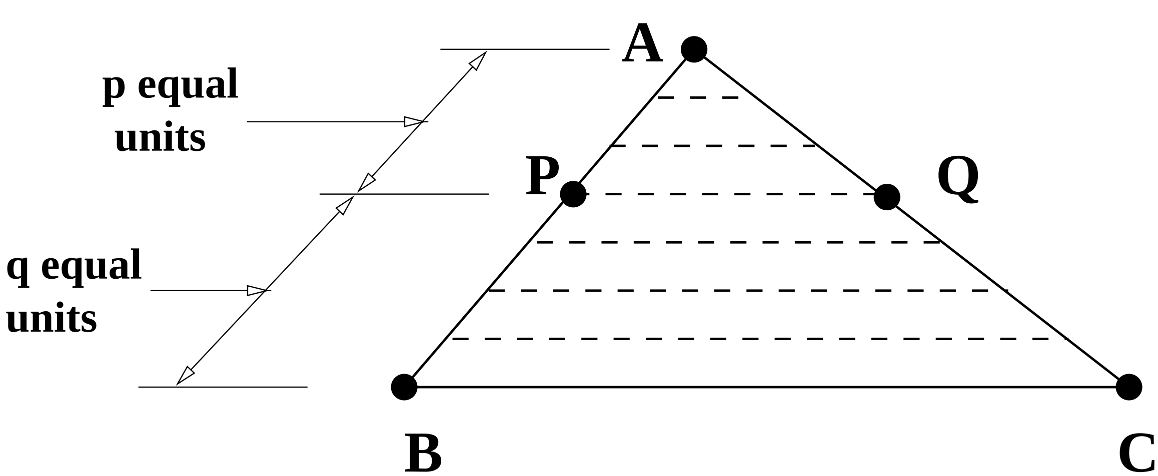


Figure 33: Proof of the ratio theorem–for a rational ratio.

If we subdivide AB into $p+q$ equal units, then AP will contain p of these units and PB will contain q of them. Draw lines as shown parallel to PQ; thus (by Theorem 5.1) all lines are parallel to the base BC (see Figure 33). By Corollary 7.3, AQ is divided into p equal portions of a new length, and QC into q such portions. Hence,

$$\frac{AQ}{QC} = \frac{p}{q} = \frac{AP}{PB} \quad .//$$

Remark. The Ratio theorem is also true when the ratio $\dfrac{AP}{PB}$ equals an *irrational* number, such as $\sqrt{2}$ or π. Since some concepts of continuity are then required, we omit the details. For an idea of the kind of axioms required here, see Axiom **O9** in Section 10.

Corollary 7.5 *In $\triangle ABC$ in Figure 32, with P on AB and Q on AC, where $PQ \| BC$, we have*

$$\frac{AP}{AB} = \frac{AQ}{AC}$$

Proof (just algebra).

$$\frac{AB}{AP} = \frac{AP+PB}{AP} = 1 + \frac{PB}{AP} = 1 + \frac{QC}{AQ} \qquad (Theorem\ 7.4)$$

$$= \frac{AQ+QC}{AQ} = \frac{AC}{AQ} \qquad //$$

8 Similar Triangles and Pythagoras' Theorem

We have available various congruence criteria: s.a.s., a.s.a., s.s.s. One condition that does not imply congruence is a.a.a. For instance, $\triangle ABC$ and $\triangle DEF$ in Figure 34 have equal corresponding angles but are clearly *not* congruent.

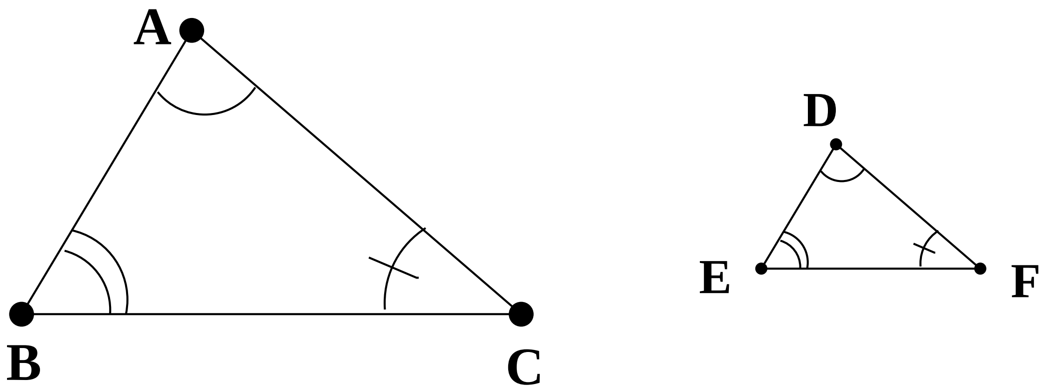


Figure 34: Similar–but not congruent–triangles.

In fact, it appears that $\triangle DEF$ is a 'scaled down' version of $\triangle ABC$. Such triangles are said to be similar:

Definition 4 *$\triangle DEF$ is* similar *to $\triangle ABC$ (written $\triangle DEF \sim \triangle ABC$) if* $\dfrac{DE}{AB} = \dfrac{DF}{AC} = \dfrac{EF}{BC}$. *That is, the three ratios of corresponding sides are equal.*

Theorem 8.1 (a.a.a. $\Rightarrow$ sim.) *If $\triangle ABC$ and $\triangle DEF$ have equal corresponding angles, then they are similar.*

Proof. We assume $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and we must prove $\dfrac{DE}{AB} = \dfrac{DF}{AC} = \dfrac{EF}{BC}$. We shall prove only the first equality since the second follows in the same way.

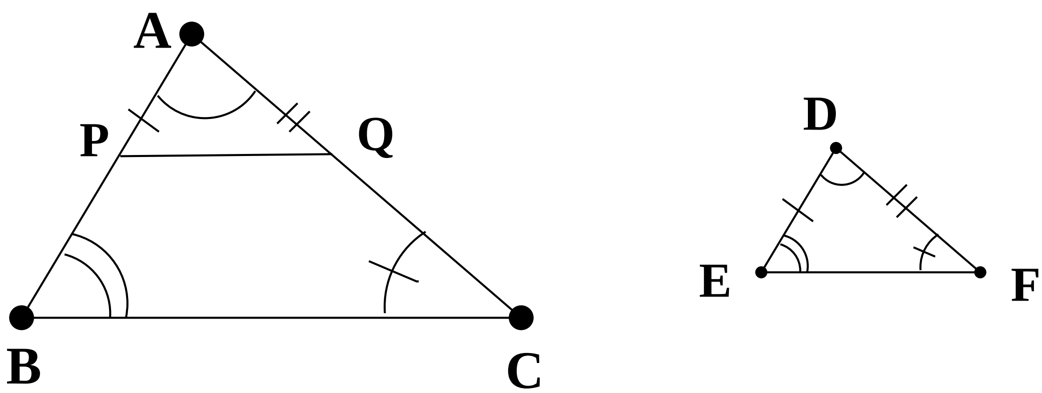


Figure 35:

In $\triangle ABC$ in Figure 35 mark off $AP = DE$ and $AQ = DF$. Thus $\triangle APQ \equiv \triangle DEF$ by (s.a.s.). Hence $\angle APQ = \angle E$. But we are given $\angle E = \angle B$, so that $\angle APQ = \angle ABC$ and $PQ \| BC$ by Corollary 5.3. Now by Corollary 7.5, $\dfrac{AP}{AB} = \dfrac{AQ}{AC}$, and hence $\dfrac{DE}{AB} = \dfrac{DF}{AC}$.

//

Theorem 8.2 (Pythagoras) *In a right triangle $\triangle ABC$, with sides a, b and hypotenuse c,*

$$a^2 + b^2 = c^2.$$

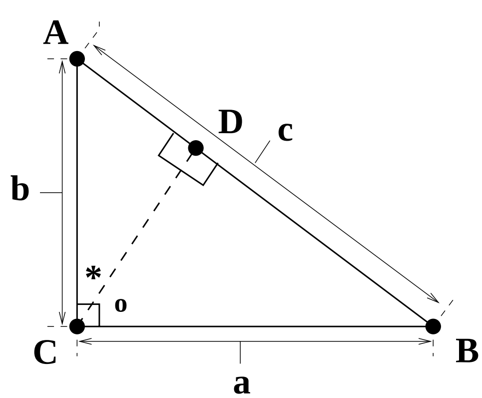


Figure 36: Pythagoras' theorem.

Proof. Draw CD perpendicular to AB. Then

$$\begin{aligned} * &= 180^\circ - \angle A - 90^\circ && \text{(by Corollary 5.4)} \\ &= 180^\circ - \angle A - \angle C \\ &= \angle B && \text{(by Corollary 5.4).} \end{aligned}$$

Thus * $= \angle B$ and similarly $\circ = \angle A$. Hence, by Theorem 8.1,

$$\triangle ADC \sim \triangle ACB \ \text{, so} \ \frac{AD}{AC} = \frac{AC}{AB} \ .$$

That is,

$$\frac{AD}{b} = \frac{b}{c} \ \text{, so} \ c(AD) = b^2 \ .$$

Likewise,

$$\triangle BDC \sim \triangle BCA \ \text{, so} \ \frac{DB}{CB} = \frac{CB}{AB} \ ,$$

and hence

$$\frac{DB}{a} = \frac{a}{c} \ \text{, whence} \ c(DB) = a^2 \ .$$

Finally,

$$a^2 + b^2 = c(DB + AD) = c(c) = c^2 \ \text{!!!} \quad //$$

9 A look back at Euclidean Geometry

9.1 Where we have been and where we are going!

We have finally reached our goal. Using only the axioms (s.a.s.) and the (Par. Ax.), along with the common sense embedded in the foundations, we have proved the basic results of Euclidean geometry.

There are now many roads open to us. Using the theorems of sections 7 and 8 we can set up x and y coordinates and the basic results of coordinate geometry (slopes, equations for lines, circles and the conic sections): see Section 23 below.

Trigonometry is little more than Theorems 8.1 and 8.2, with a bag of algebraic tricks telling us how to manipulate trigonometric functions and identities.

You are also equipped now to learn a lot of unusual and pretty mathematics. I recommend [6]; it has lot's of challenging and beautiful geometrical ideas.

In this course, we shall eventually encounter the ideas of **isometry** and **group**, and their applications to patterns and designs. First, however, we digress a bit in the next section (§ 10) to explore a bit more the rather technical axioms which underly our common sense foundations for geometry.

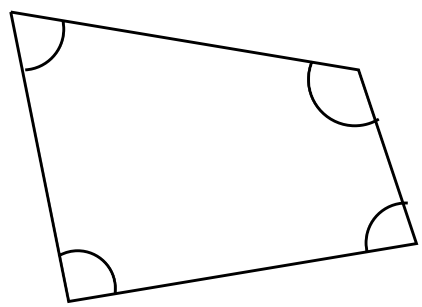
But before that you should try lots of problems to exercise your geometrical muscles. Several subsections of general Euclidean problems follow. (See § 4.9 for some basic advice.)

Here, however, you are not constrained by axioms, so you may use all standard results of Euclidean geometry – but you should supply a reference for anything 'obscure'.

9.2 Some Introductory Problems

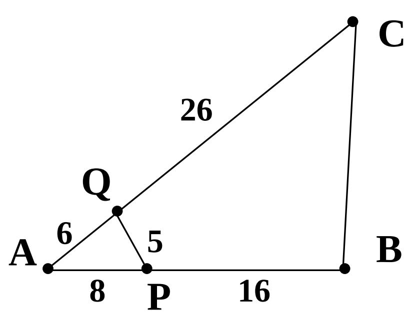
Here are a few basic problems in Euclidean geometry. You may use all standard results in Euclidean geometry, including those covered up to § 8 in the text.

1. (a) Prove that the angles of a quadrilateral sum to 360°.

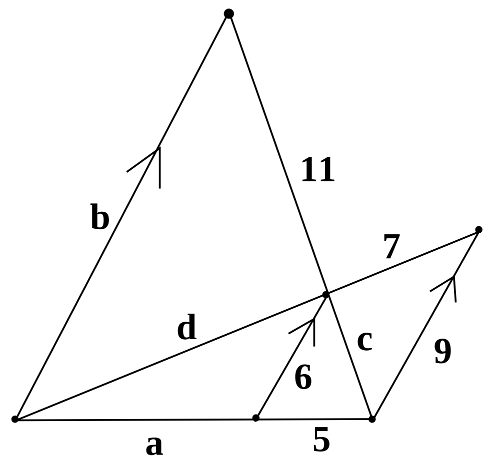


   (b) What is the sum of the interior angles in an n-gon ?

2. (a) Find BC in:



(b) Arrows indicate parallel lines. Find a, b, c, d.

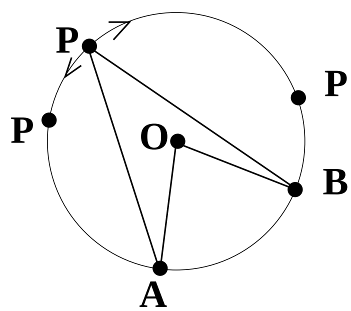


3. **Results on Circles**.

Prove the following results, which are mainly concerned with circles.

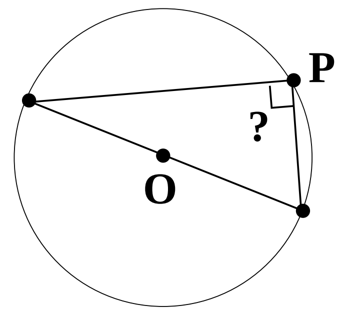
(a) Given points A, B, and P on a circle with centre O, show that no matter where P is positioned on the upper arc from A to B, we have

$$\angle AOB = 2\angle APB \ .$$



(Careful – there are two slightly different cases here.)

(b) The angle in a semicircle is a right angle:



(c) In a right triangle let the altitude h_c to the hypotenuse cut the hypotenuse into two parts of length c_1 and c_2. Show that $h_c = \sqrt{c_1 c_2}$.

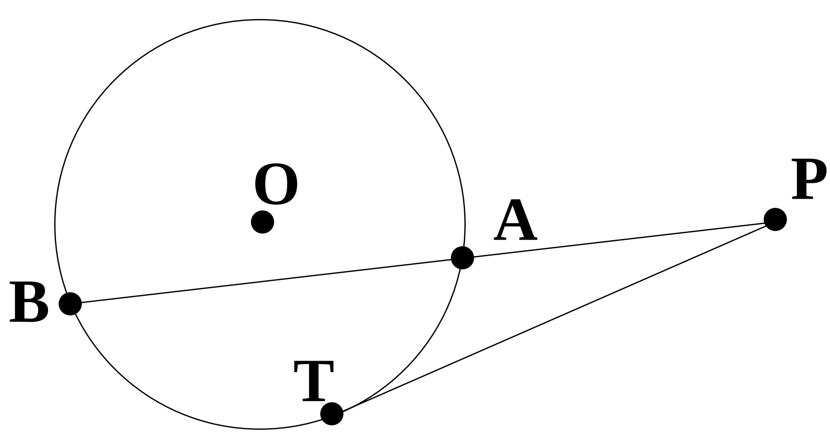
(d) Given a segment of length 1, provide a R-C construction for $\sqrt{6}$. (Hint—try to use the previous two parts.)

(e) The opposite angles of a cyclic quadrilateral are supplementary. (Hint—look up definitions for cyclic quadrilateral, supplementary.)

(f) For a point P external to a circle give a R-C construction for the two tangents from P to the circle. (Hint—again part 3b will be useful.)

(g) For an external point P let the segment PT be tangent to a circle at T and let another line PAB cut the circle at A and B. Show that

$$(PT)^2 = (PA)(PB).$$



(h) Let two chords AB and CD of a circle intersect at F. Show that

$(AF)(FB) = (CF)(FD)$.

4. We've never used areas in proof, since we haven't defined *area*! You all 'know' the area of $\triangle ABC$ is

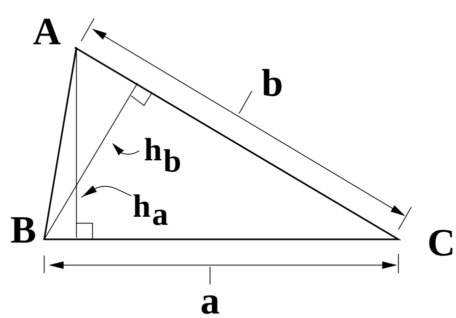
$$\frac{\text{base} \cdot \text{height}}{2} \quad .$$

But why don't you get a different area using another side for the base? Here's the answer.

In $\triangle ABC$, let h_a and h_b be the altitudes to sides a and b respectively. Prove that

$$ah_a = bh_b.$$

(Hint: There are two similar right triangles sharing $\angle C$.)



5. Prove the Law of Sines for $\triangle ABC$:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

(Hint: This follows easily from question 4 above.)

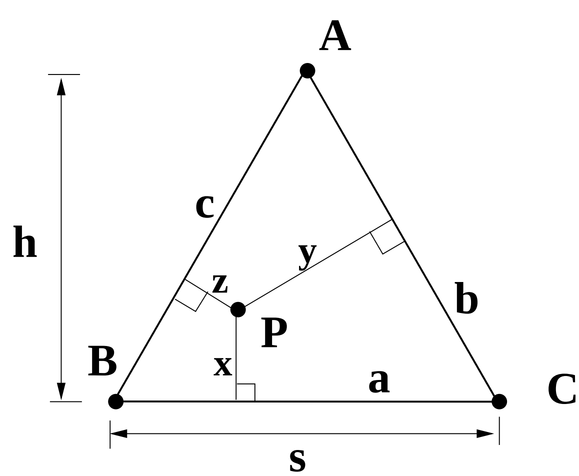
6. Facing a slightly steamed-over mirror, hold one eye shut and trace the outline of your face in the mirror. Explain why the outline is exactly 1/2 the width and 1/2 the height of your face (see [18, p.138]).

9.3 Some Deeper Problems

Some of these problems explore new territory or have an extra element of trickiness.

1. *Trilinear Coordinates* (See [6, p.89].)

   Let $\triangle ABC$ be equilateral with side s and altitude h. For any point P in the plane let x, y, z be the distances of p from sides a, b, c respectively. We take x as negative if P lies on the *other* side of side a from A, and similarly for y and z. The triple (x, y, z) is the set of trilinear coordinates for P. (They are quite different from ordinary Cartesian coordinates).

   (a) Find $\angle A, \angle B, \angle C$. Give h in terms of s.

   (b) Find a constant k such that for *all* points P,

   $$x + y + z = k \quad .$$

   (These coordinates are *redundant*. To verify this equation, compute areas of $\triangle PBC, \triangle PCA, \triangle PAB$.)

   (c) Extend sides a, b, c infinitely far. Into how many regions is the plane decomposed? Label each region $(+++)$, $(++-)$, etc. according to the signs of x, y, z.

(d) Give coordinates for A, B, C. Give equations for lines a, b, c.

(e) Remember that the equation in part (b) always holds. What loci are described by the equations:

(i) $x + y = \frac{h}{2}$?

(ii) $x = y$?

(iii) $x^2 + y^2 + z^2 = h^2$? (tricky!)

(f) Suppose $h = 1$. Using a sketch describe all the solutions to the equation

$$x + y + z = 1 \ ,$$

where x, y, z are *integers*. (This is a *Diophantine equation*.)

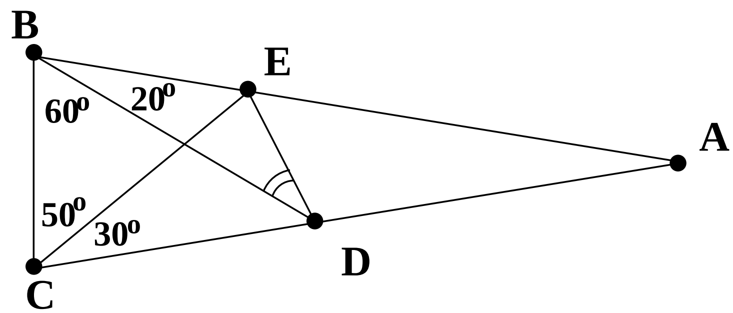
2. (Some Problems Adapted from *Geometry Revisited* (reference [6])).

(a) Suppose $\triangle ABC$ and $\triangle A'B'C'$ have sides respectively parallel ($AB \parallel A'B'$, etc.) Show that the lines AA', BB', CC' (extended) are either concurrent or parallel.

(b) Find the length of the internal bisector of the $90°$ angle in a triangle with sides $3, 4, 5$.

(c) An isosceles $\triangle PAB$ with base angles $15°$ at A, B is drawn inside square $ABCD$. Prove $\triangle PCD$ is equilateral.

(d) Find $\angle EDB$ in

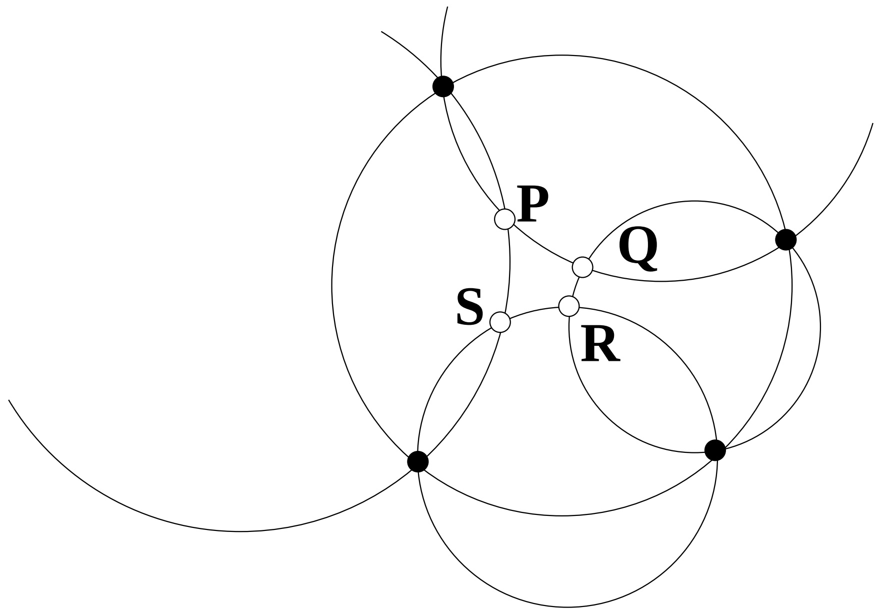


(e) Suppose PT and PU are tangents from P to two concentric circles, with T on the smaller, and let segment PT meet the larger circle at Q.

Show $PT^2 - PU^2 = (QT)^2$.

(f) You are at the top of a road-side tower $50m$. high on the prairies. Just where the straight road vanishes in the distance you see an elevator. You drive to the elevator finding it to be 25.2 km. away. What is the radius of the earth?

3. If five circular arcs intersect as shown below, prove that a sixth circle can be drawn passing through P, Q, R, S:



9.4 General Problems.

There is an inexhaustible supply of general problems in Euclidean geometry. Try as many from the following collection as you can. In each problem you may use all standard results in Euclidean geometry, including those covered up to § 8 in the text.

1. If the bisector of an exterior angle of a triangle is parallel to the opposite side, the triangle must have two of its angles equal.

2. The bisector of the exterior angle at one vertex of a triangle cannot be parallel to the bisector of either of the angles at the other vertices.

3. P and Q are the centres of two circles each of which lies entirely outside the other. PM and QN are two parallel radii which are so placed that MN meets the circles again at X and Y. Prove $PX||QY$.

4. The medians BD and CE of $\triangle ABC$ are produced to X and Y respectively so that $BD = DX$ and $CE = EY$. Prove that X, A and Y are in a straight line.

5. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.

6. In quadrilateral $ABCD$, $\angle A = \angle D$ and $\angle B = \angle C$. Prove $AD||BC$.

7. P is the midpoint of LM in $\triangle KLM$. Prove that if $PL = PK = PM$, $\angle LKM$ is a right angle.

8. The sum of the exterior angles at two opposite vertices of any quadrilateral is equal to the sum of the interior angles at the other two vertices.

9. PQR is a triangle in which $PQ = PR$. PQ is produced to S so that $QS = QR$. Prove $\angle PRS = 3$ times $\angle QSR$.

10. In $\triangle ABC$, A is a right angle. The bisectors of the angles at B and C meet at D. Prove that $\angle BDC$ contains $135°$.

11. If in $\triangle ABC$, A contains x degrees, and the bisectors of the angles at B and C meet at D, show that $\angle BDC = \left(90 + \frac{x}{2}\right)^\circ$.

12. The bisectors of the exterior angles at B and C in $\triangle ABC$ meet at D. Show that if $\angle A$ contains x degrees, $\angle BCD = \left(90 - \frac{x}{2}\right)^\circ$.

13. The bisector of the interior angle at A and the bisector of the exterior angle at B in $\triangle ABC$ meet at P. Prove that $\angle APB = \frac{1}{2}\angle C$.

******

14. Consider any $\angle ABC$. P is any point on BD, the bisector of $\angle ABC$. PX is drawn parallel to BC and meets AB at X. Prove that $XP = XB$.

15. $\triangle PQR$ is a triangle in which $\angle Q = \angle R =$ twice $\angle P$. The bisector of $\angle Q$ meets PR at M. Prove that $PM = MQ = QR$.

16. Suppose $DE = DF$ in the isosceles $\triangle DEF$.a A line drawn perpendicular to EF cuts DE at X and FD (produced) at Y. Prove that $DX = DY$.

17. In $\triangle ABC$, $AB > AC$. D is a point on AB such that $AD = AC$. Prove that $\angle DCB$ is equal to one-half the difference between $\angle ACB$ and $\angle ABC$.

18. $PQRS$ is a parallelogram. X and Y are the midpoints of PQ and RS respectively. Prove that PR and XY bisect each other.

19. $ABCD$ and $ABXY$ are any two parallelograms having a common side AB. Prove that $CDYX$ is a parallelogram.

20. If in quadrilateral $KLMN$, KL and MN are parallel, and KN and LM are equal but not parallel, prove that $\angle K = \angle L$ and $\angle M = \angle N$.

21. In parallelogram $PQRS$, X and Y are the midpoints of PS and QR respectively. Prove that PY and XR trisect QS.

22. Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the midpoints of the other two sides.

23. In any quadrilateral the midpoints of the sides are the vertices of a parallelogram.

24. In any quadrilateral the midpoints of two opposite sides and the midpoints of the diagonals are the vertices of a parallelogram.

25. The straight lines which join the midpoints of opposite sides of any quadrilateral, and the straight line which joins the midpoints of its diagonals, all pass through one point.

26. The internal and external bisectors of $\angle BAC$ meet a line through C, and parallel to AB, in the points P and Q. Prove that $PC = CQ$.

27. Two equal straight lines AC, BD bisect each other. Show that the quadrilateral $ABCD$ is a rectangle.

28. $ABCD$ is a square and on the diagonal AC , the segment AE is cut off equal to AB; through E , FEG is drawn perpendicular to AC, meeting BC in F and CD in G. Show that $\angle FAG$ is half a right angle.

29. $\triangle ABC$ is an equilateral triangle and D is any point on AB; on the side of AD remote from C an equilateral triangle $\triangle ADE$ is constructed; prove that $BE = CD$.

30. L, M, N are the midpoints of the sides BC, CA, AB of $\triangle ABC$. BM cuts LN in X and CN cuts LM in Y. Prove that $4(XY) = BC$.

******

31. In a right-angled triangle the hypotenuse is double the median drawn from the vertex of the right angle.

32. If from the vertex of a triangle two straight lines be drawn, one perpendicular to the base, and the other bisecting the vertical angle, the angle they contain is equal to one-half the difference between the base angles of the triangle.

33. ABC is a triangle. AK and AL are drawn perpendicular to BK, CL the bisectors of the exterior angles at B and C respectively. Prove that $KL || BC$.

34. The sum of the perpendiculars drawn from any point within an equilateral triangle to the three sides, is equal to the perpendicular drawn from the vertex to the base.

35. AB is a given straight line of unlimited length; C and D are two points on the same side of AB. Find a point P in AB such that $\angle CPA = \angle DPB$.

36. $ABCDE$ is a regular pentagon: AC, BE cut each other in F. Prove that $\angle CBF = \angle CFB$.

37. Show that if two parallelograms have a common diagonal, the other angular points are at the corners of another parallelogram.

38. Show that if one pair of opposite sides of a quadrilateral are equal, the midpoints of the other two sides and the midpoints of the diagonals are the vertices of a rhombus.

39. Suppose $\angle B = 90°$ in $\triangle ABC$. On AB, BC, respectively, points X and Y are taken. Prove that $AY^2 + CX^2 = AC^2 + XY^2$.

40. $ABCD$ is a rectangle and O is any point. Prove that $OA^2 + OC^2 = OB^2 + OD^2$. Show that this is true even when O is not in the same plane as the rectangle.

41. In $\triangle DEF$, DX is drawn perpendicular to EF. Prove that $DE^2 - DF^2 = EX^2 - XF^2$.

42. From a point O inside the $\triangle ABC$, perpendiculars OD, OE, OF are drawn to BC, CA, AB respectively. Prove that $BD^2 + CE^2 + AF^2 = BF^2 + AE^2 + CD^2$.

43. In triangle $\triangle ABC$ suppose that AN, the perpendicular from A to BC, falls within the triangle. If $BN \cdot NC = AN^2$, prove that $\angle BAC$ is a right angle.

44. $\triangle ABC$ is any triangle. D is the foot of the perpendicular from A to BC. BH is drawn perpendicular to AB and equal to CD; and CK is drawn perpendicular to AC and equal to BD. Prove that $AH = AK$.

9.5 Circles and Other General Problems.

Circles are endlessly fascinating objects, since they have so many unexpected properties. This large collection of exercises is mainly concerned with circles.

Again you may use all standard results in Euclidean geometry, up to and including those in § 8 of the Notes.

1. **Theorem.** Of two chords in a circle, the one which is nearer the centre is longer. (Hint: Use Pythagoras twice.)

2. Two circles cut in A and B. Through A and B are drawn parallel lines XAP and YBQ, meeting the circles at X and P, and Y and Q respectively. Prove $XAP = YBQ$.

3. Two circles with centres C and D cut in A and B. Through A is drawn a line cutting the circles again in X and P. The lines CX and PD cut in Y. Prove $\angle CAD = \angle XYP$.

4. Two circles cut in A and B. C is the mid-point of the line joining their centres. The line through A perpendicular to CA cuts the circles again in X, P. Prove $AX = AP$.

5. Prove that the angle in a major segment of a circle is acute, and the angle in a minor segment is obtuse. (Be sure you understand the terminology here before proceeding!)

6. A quadrilateral is inscribed in a circle. Prove that the sum of either pair of opposite angles equals two right angles.

7. $ABCD$ is a quadrilateral inscribed in a circle; AB and CD are each equal to the radius. AC and BD meet in E. Find the number of degrees in $\angle AEB$.

8. $ABCD$ is a quadrilateral inscribed in a circle whose centre is O. AC and BD intersect at E. Prove that $\angle AOB + \angle COD = 2\angle AEB$.

9. $\triangle ABC$ is a triangle inscribed in a circle. The bisector of $\angle A$ meets BC at D and the circle again at E. Prove that $\triangle ADB$ is equiangular to $\triangle ACE$.

10. D and E are the midpoints of the equal sides AB, AC of an isosceles $\triangle ABC$. Prove D, B, C, E are concyclic.

11. In $\triangle KLM$, LX and MY are drawn perpendicular to KM, KL respectively. Prove that:

    (a) Y, L, M, X are concyclic.

    (b) $\angle LMY = \angle LXY$, and $\angle MLX = \angle MYX$.

12. $ABCD$ is a parallelogram. A circle with centre A and radius AD cuts DC, produced if necessary, in E. Prove that A, B, C and E are concyclic.

13. Two circles intersect at A and B. Any straight line CD is drawn through A and terminated by the circumferences at C and D. The bisector of $\angle CBD$ meets CD in P. Prove that $\angle APB$ is constant (given that CD *is* variable).

14. Prove that the locus of the midpoints of a set of parallel chords of a circle is a diameter.

15. In $\triangle ABC$, BC is fixed in length and position and $\angle A$ is constant. The bisectors of angles B and C meet at P. What is the locus of P?

16. Chords of a given circle are drawn through a given point. Prove that the locus of their midpoints is a circle.

17. Two circles intersect at A and B. AP and AQ are diameters. Prove PBQ is a straight line.

18. AB is the diameter of a circle. With centre B and radius AB, a second circle is drawn. Prove that any chord of the second circle through A is bisected by the circumference of the first.

19. In $\triangle ABC$, perpendiculars BD, CE are drawn to AC, AB respectively. Prove that the midpoint of BC is equidistant from D and E.

20. P is any point on the arc of a semicircle of which AB is the diameter. PQ is drawn perpendicular to AB. Prove that $\triangle PQA$, $\triangle PQB$ and $\triangle PAB$ are equiangular.

21. Show how the square corner of a sheet of paper may be used to locate a diameter of a circle whose centre is unknown.

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22. $PQRS$ is a cyclic quadrilateral. PS and QR, when produced, intersect in O. Prove $\triangle OPQ$ is equiangular to $\triangle ORS$.

23. If a triangle be inscribed in a circle and an angle be taken in each of the three circular segments outside the triangle, the sum of these angles is four right angles.

24. $ABCD$ is a quadrilateral inscribed in a circle and $AD||BC$. Prove $\angle B = \angle C$ and $AC = BD$.

25. Two circles intersect in A and B. PAR and QBS are straight lines terminated by the circumferences. Prove $PQ||RS$.

26. Any parallelogram inscribed in a circle is a rectangle.

27. Through a point on the diagonal of a square, lines PR, QS are drawn parallel to the sides, P, Q, R, S being on the sides. Prove that these four points are concyclic.

28. O is the centre of a circle, CD is a diameter, and AB a chord perpendicular to CD. If B is joined to any point E on CD and BE produced to meet the circle again in F, then A, O, E, F are concyclic.

29. P is on the bisector of the $\angle BAC$ of $\triangle ABC$; circles APB and ACP cut BC in Q and R respectively. Prove $PQ = PR$.

30. In $\triangle ABC$, $\angle A$ is a right angle. AD is drawn perpendicular to BC. Prove that AC is a tangent to the circle ABD.

31. PQ is a diameter of a circle whose centre is O; R is taken on the tangent at Q such that $QR = PQ$. If PR cuts the circle at S, prove that $PS = SQ = SR$.

32. O is the centre of a circle: C is any point on a tangent which touches the circle at A: CO cuts the circle at B and AD is perpendicular to OC. Prove that AB bisects $\angle DAC$.

33. Tangents are drawn to a given circle from an external point A and touch the circle at B and C. On the arc BC, nearer to A, any point P is taken and a tangent is drawn at P to meet AB, AC in X, Y respectively. Prove that the perimeter of $\triangle AXY = AB + AC$. (Thus the perimeter is constant even as P varies.)

34. A circle inscribed in $\triangle ABC$ touches BC, CA, AB in X, Y, Z respectively. Prove that $AZ + BX + CY$ is equal to one-half the perimeter of $\triangle ABC$.

35. Two parallel tangents to a circle, meet a third tangent at P and Q. Prove that PQ subtends a right angle at the centre.

36. $ABCD$ is a quadrilateral circumscribed about a circle with centre O. Prove that:

(a) $AB + CD = AD + BC$.

(b) $\angle AOB + \angle COD = 180°$.

37. A is the centre of a circle and B is any point on the circumference: AB is produced to P so that $BP = AB$: tangents PQ, PR are drawn to touch the circle at Q, R. Prove that PQR is an equilateral triangle.

38. Two circles cut in X and Y. A line through X cuts them in A and B respectively. AP and BQ are parallel chords, one of each circle. Prove that P, Y, Q are collinear.

39. Find the angle between the tangents to a circle from a point whose distance from the centre is equal to the diameter.

40. From a point P, two tangents PA, PB are drawn to a circle ABD of which the centre is O. The chord AB joins the points of contact and from A a diameter AOD is drawn. Show that the angle $\angle APB$ is double $\angle BAD$.

41. AD is perpendicular to the base BC of $\triangle ABC$; AE is a diameter of the circumscribing circle. Prove that $\triangle ABD$ is equiangular to $\triangle AEC$.

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42. If a chord AB of a circle ABC is parallel to the tangent at C, prove that $AC = BC$.

43. AB, AC are chords of a circle ABC. AT is a tangent. Prove that if AB bisects $\angle TAC$, $AB = BC$.

44. Two circles intersect at A and B and through any point P on the circumference of either one of them, straight lines PAC, PBD are drawn to cut the other circle at C and D. Show that CD is parallel to the tangent at P.

45. The tangent at a point P on a circle, and a chord AB are produced to meet at Q. Prove $\angle Q = \angle PBA - \angle PAB$.

46. Two circles touch each other at A: through A any straight line is drawn cutting the circumferences again at P and Q. Prove that radii through P and Q are parallel.

47. Two circles ACO, BDO touch each other and AOB, COD are straight lines. Show that AC is parallel to BD.

48. If two parallel diameters be drawn in two circles which touch each other, the point of contact and an extremity of each diameter are in the same straight line.

49. Two circles with centres A, B touch externally at T: a circle touching AB at T cuts the two original circles in P and Q respectively. Prove that AP, BQ are tangents to the new circle.

50. Two circles whose centres are A and B touch externally at C. A common tangent touches the former circle at P and the latter at Q, and meets the tangent at C in R. AR and PC meet in S, and BR and QC meet in T. Show that $RSCT$ is a rectangle.

51. $\triangle ABC$ is an acute-angled triangle with AB equal to AC. In AB a point D is taken so that $CD = CB$. Prove that the circle circumscribing $\triangle ADC$ touches BC.

52. Two circles intersect at X and Y: through X any straight line is drawn cutting the circles at L and M. The tangents at L and M intersect at N. Prove that $YLNM$ is a cyclic quadrilateral.

53. T is any point outside a circle ABC whose centre is the point O. Through T two lines are drawn, TA touching the circle and TBC cutting it. If M is the midpoint of BC, show that $\angle AMT = \angle AOT$.

54. Two circles intersect in A and B: PQ is a common tangent. Prove that the angles $\angle PAQ$ and $\angle PBQ$ are supplementary.

55. Consider any $\triangle ABC$ and suppose DE is parallel to BC and cuts the sides in D and E. Prove that the circumcircles of the triangles $\triangle ABC$ and $\triangle ADE$ touch at A.

56. $ABCD$ is a cyclic quadrilateral whose diagonals intersect at E. A circle is drawn through A, B and E. Prove that the tangent to this circle at E is parallel to CD.

57. Suppose $\triangle ABC$ is inscribed in a circle and that the tangents at B and C meet in T. Prove that if through T a straight line is drawn parallel to the tangent at A, meeting AB, AC (produced) in F and G, then T is the midpoint of FG.

58. AB is a fixed chord in a given circle. P is any point in the major arc. Find the locus of the centre of the inscribed circle of $\triangle PAB$.

59. In $\triangle ABC$ suppose D, E and F are any points on the sides BC, CA and AB respectively. Prove that the circles AFE, BFD and CDE have one common point.

60. ANC, BND are chords of a circle; the tangents at A and B meet at P; the tangents at C and D meet at Q. Prove that the sum of the angles $\angle P$ and $\angle Q$ is twice $\angle BNC$.

61. $ABCD$ is a cyclic quadrilateral; DE is a chord bisecting the angle between CD produced and BD. Prove that AE (produced if necessary) bisects the angle between BA produced and AC.

62. If two chords of a circle intersect at right angles, the sum of the squares of their lengths is equal to the square of the diameter.

63. If the sum of one pair of opposite sides of a quadrilateral be equal to the sum of the other pair, a circle may be inscribed in the quadrilateral.

64. D, E, F are the points of contact of the sides BC, CA, AB of $\triangle ABC$, with its inscribed circle; also FK is perpendicular to DE and EH is perpendicular to FD. Prove that $HK \parallel BC$.

65. Parallel chords AC, BD of a circle are drawn through the ends of a diameter AB. Prove that CD is also a diameter of the circle.

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66. The straight line drawn from the mid-point of one side of a triangle, parallel to a second side, bisects the third side.

67. $ABCD$ is a trapezium in which $AD \parallel BC$. Prove that the straight line through E (the midpoint of AB) and parallel to BC, bisects CD.

68. L is any point in the side DE of $\triangle DEF$. From L a line drawn parallel to EF meets DF at M. From F a line drawn parallel to ME meets DE produced at N. Prove that $\dfrac{DL}{DE} = \dfrac{DE}{DN}$.

69. On the sides BC, CA of $\triangle ABC$ the points D, E are taken respectively such that $CD = 2(BD)$ and $CE = 2(EA)$. The lines AD, BE intersect at O, and CO is produced to cut AB in K. Show that $AK = KB$, $CO = 4(OK)$ and $2(BO) = 3(OE)$.

70. D and E are points on the sides BC, CA respectively of $\triangle ABC$ such that $BD = \dfrac{1}{2}\,DC$ and $CE = EA$. Show that AD bisects BE.

71. From any point O on the diagonal AC of the quadrilateral $ABCD$ lines OX, OY are drawn parallel to AB, AD respectively, so as to meet CB, CD respectively in X, Y. Show that XY is parallel to BD.

72. $ABCD$ is any quadrilateral. From P, any point on BC, the line PR is drawn parallel to BA to meet AC in R; and PQ is drawn parallel to BD to meet DC in Q. Prove that $RQ \parallel AD$.

73. AB and CD are the parallel sides of a trapezium $ABCD$ whose diagonals intersect at O. Prove that $\triangle OAB \sim \triangle OCD$ and write down the equal ratios of corresponding sides.

74. The medians BE, CF of $\triangle ABC$ intersect at G. Prove that $BG = 2(GE)$ and $CG = 2(GF)$.

75. $ABCD$ is a parallelogram; a straight line is drawn through A meeting BD at E, BC at F and DC produced at G. Prove that $\frac{AE}{EF} = \frac{AG}{AF}$.

76. In quadrilateral $ABCD$, AC and BD intersect at O. Prove that if $\frac{AO}{OG} = \frac{BO}{OD}$, then $AB \parallel CD$.

77. In $\triangle ABC$, AD is drawn perpendicular to BC, and $\frac{BD}{AD} = \frac{AD}{DC}$. Prove that $\angle BAC$ a right angle.

78. E is any point on a radius OD of a circle with centre O. F is taken in OD produced such that $\frac{OE}{OD} = \frac{OD}{OF}$. P is any point on the circumference. Prove that $\triangle OPE \sim \triangle OPF$ and that PD bisects $\angle EPF$.

79. $ABCD$ is a quadrilateral. On the side of AB remote from C, $\angle BAE$ is made equal to $\angle CAD$, and $\angle ABE = \angle ADC$. Prove $\angle ECA = \angle BDA$.

80. C is any point on the diameter AB of a semicircle; the perpendicular to AB from C cuts the semicircle at D and the chord AF in E. Prove

$$AE \cdot AF = AC \cdot AB = AD^2 \ .$$

81. AB is a diameter of a circle of radius r. A tangent at a point T on the circle cuts the (other) tangents at A and B in C, D respectively. Prove $AC \cdot BD = r^2$.

82. In $\triangle ABC$, $\angle B = \angle C = 2\angle A$ and CD, the bisector of $\angle C$, meets AB in D. Prove that $\dfrac{AB^2}{BC^2} = \dfrac{AB}{BD}$.

83. In $\triangle ABC$ suppose AB is double BC. Also suppsoe E is a point on AB such that EB is half BC. Prove that $\angle BCE = \angle CAB$.

84. S is a point in the side PQ of $\triangle PQR$: ST is drawn parallel to QR and of such a length that $ST : QR = PS : PQ$. Prove that T lies in PR. (hint: first show $\angle SPT = \angle QPR$.)

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85. Through any point P on the common chord MN of two intersecting circles, lines APB, CPD are drawn, one of them meeting the circumference of one circle in A, B, and the other meeting the circumference of the second circle in C and D. Prove that

$$PA \cdot PB = PC \cdot PD$$

.

86. Through points P and Q on a circle, straight lines APB, CQD are drawn meeting a concentric circle in A, B, and C, D respectively. Prove that

$$AP \cdot PB = CQ \cdot QD \quad .$$

87. Two chords AB, CD of a circle intersect at an internal point X. Prove

$$AB^2 + XC^2 + XD^2 = CD^2 + XA^2 + XB^2 \quad .$$

88. LM is a chord of a circle and it is bisected at K. DKJ is another chord. On DJ as diameter a semicircle is drawn and $KS \perp DJ$ meets the semicircle at S. Prove that $KS = KL$.

89. If two circles intersect, the tangents drawn to them from any point on the common chord produced, are equal.

90. If two circles intersect, their common chord bisects their common tangents.

91. Suppose $\triangle ABC$ is right-angled at C. From any point D on AB, DE is drawn perpendicular to AB, meeting AC at E. Prove

$$AC \cdot AE = AB \cdot AD \ .$$

92. From any point P outside a circle whose centre is O, any secant PAB is drawn cutting the circle at A and B. Prove that

$$PA \cdot PB = OP^2 - OA^2 \ .$$

93. The common chord of two intersecting circles is produced to a point A. From A, two lines are drawn, one to cut one of the circles at B and C and the other to cut the second circle at D and E. Show that B, C, E and D are concyclic.

94. The bisectors of the interior and exterior angles at any vertex of a triangle, divide the opposite side internally and externally in the same ratio.

95. If the bisectors of the angles E and F of $\triangle DEF$ divide DF and DE proportionally, then $DE = DF$.

96. PX is a median of $\triangle PQR$. The bisectors of angles $\angle PXQ$ and $\angle PXR$ meet PQ, PR at M and N respectively. Prove that $MN \parallel QR$.

97. In $\triangle ABC$, $DE \parallel BC$ and meets AB, AC in D and E respectively. The bisector of $\angle A$ cuts DE at F and BC at G. Prove that $\frac{BG}{GC} = \frac{BD}{CE} = \frac{DF}{FE}$.

10 The Axiomatic Foundations for Geometry

10.1 Euclid's Axioms

We shall discuss here the history and significance of Euclid's remarkable attempt to put geometry on solid axiomatic ground.

Modern criticisms (and repair) of Euclids attempt concern mostly very subtle, though important, matters. To get some idea of Euclid's approach, here is a glimpse of the first few pages of the Elements [8][vol.1, pages 153ff]:

BOOK 1

Definitions.

1. A **point** is that which has no part.

2. A **line** is breadthless length.

3. The extremities of a line are points.

4. A **straight line** is a line which lies evenly with the points on itself.

• • •

8. A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

9. And when the lines containing the angle are straight, the angle is called **rectilineal**.

10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.

• • •

15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.

• • •

23. **Parallel** straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

By modern standards of logic, there is much to object to here. But to be fair, Euclid's main goal was surely to educate his students. Perhaps the above 'definitions' were meant merely to bolster ones intuition.

Besides the above primitive terms and definitions, we need axioms. These Euclid divided into two groups (Postulates and Common Notions), presumably for rather subtle philosophical reasons. Perhaps the Postulates were to include only those basic statements particular to geometry, thus leaving the Common Notions for use in other sciences. Or conceivably the distinction was interpolated into the text by one of the innumerable Greek, Arabic and Latin scribes who later translated The Elements.

Again Euclid's intent was to teach the subject, so there is perhaps no point debating the distinction. Modern mathematicians make do with one set of axioms and the basic rules of inference. Here, in fact, is what Euclid wrote:

POSTULATES

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

COMMON NOTIONS.

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

Postulate 5, which ultimately deals with parallelism, is the crucial axiom. Certainly it is aesthetically less satisfying, so much so that mathematicians through the ages have tried to deduce Postulate 5 from the remaining four.

These efforts were doomed to failure, since Bolyai and Lobachevskii showed in the 19th century that the negation of Postulate 5 is consistent with 1 to 4. We thus are led to the birth of *non-Euclidean geometry*.

By way of conclusion, here is how a typical proposition (s.a.s.) appears in the definitive modern translation of The Elements [8]:

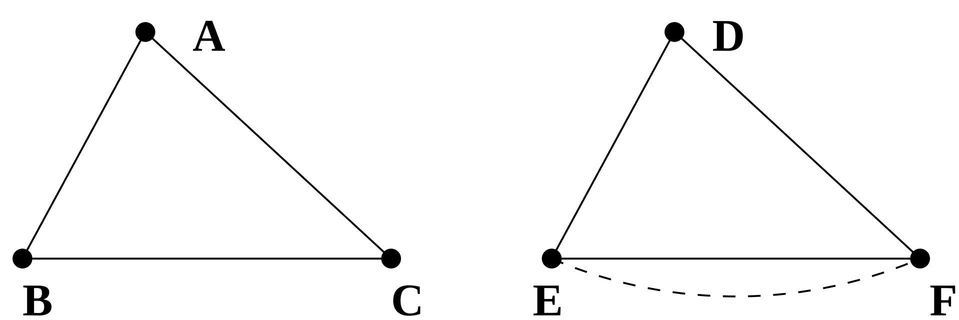
PROPOSITION 4.

If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF respectively, namely AB to DE and AC to DF, and the angle BAC equal to the angle EDF.

I say that the base BC is also equal to the base EF, the triangle ABC will be equal to the triangle DEF, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend, that is, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

For, if the triangle ABC be applied to the triangle DEF, and if the point A be placed on the point D and the straight line AB on DE, then the point B will also coincide with E, because AB is equal to DE. etc.



The 'proof', though intuitively clear, is not a proof at all, since the argument makes hidden use of a technique which is logically equivalent to what should be proved. This is the cardinal sin of assuming what is to be proved.

In fact, something like Proposition 4 must be taken as an axiom (such as **C5** in Veblen's approach outlined below).

10.2 Modern Foundations for Geometry

Several mathematicians of the late 19th and early 20th centuries have provided rigorous foundations for ordinary geometry. We mention Hilbert, Pasch, Peano and Veblen among others. Consequently, there are many logical approaches to the same geometrical destination.

In Pasch's development of ordered geometry, as simplified by Veblen, the only primitive concepts are **points** A, B, ... and the relation of **intermediacy** [ABC], which says that B is between A and C. If B is not between A and C, we say simply "not [ABC]." There are altogether 15 axioms (see below: **O1 - O9, C1 - C5, Par**). Of course, some earlier axioms are used to prove theorems which must be established before later axioms make sense. Likewise, various definitions can be made only at certain points in the story.

At least once in your mathematical career you should work through the details yourself. I recommend the beautifully written treatment in H. S. M. Coxeter's *Introduction to Geometry* [3]:

(a) §12.1, 12.2, 12.4, 12.5, 12.6

(b) §15.1, 15.2, 16.1

At this point you will have the groundwork needed for high school geometry. Coxeter further develops the material as follows, with many elegant mathematical excursions.

(c) *For Euclidean Geometry*: return to 13.1, 13.2, 13.3, 13.4, 13.6, 13.7. Or consult [21, 10] for more detail on foundations, or [8] for Euclid in the original.

(d) *For Non-Euclidean Geometry*: 16.2, 16.3, 16.4, 16.5, 16.6, 16.7, 16.8 (and perhaps 20.1, 20.2, 20.3, 20.4, 20.5, 20.6).

The Axioms

Order Axioms

O1: There are at least two points.

O2: If A and B are two distinct points, there is at least one point C for which $[ABC]$.

O3: If $[ABC]$, then A and C are distinct : $A \neq C$.

O4: If $[ABC]$, then $[CBA]$ but not $[BCA]$.

Definitions. If A and B are two distinct points, the **segment** AB is the set of points P for which $[APB]$. We say that such a point P is **on** the segment. Later we shall apply the same preposition to other sets, such as 'lines.'

The **interval** $\overline{AB}$ is the segment AB plus its **end points** A and B:

$$\overline{AB} = A + AB + B.$$

The **ray** A/B (' starting at A, away from B') is the set of points P for which $[PAB]$.

The **line** AB is the interval $\overline{AB}$ plus the two rays A/B and B/A:

$$\textit{line } AB = A/B + \overline{AB} + B/A.$$

Note that

$$\textit{Interval } \overline{AB} = \textit{interval } \overline{BA};\ \textit{line } AB = \textit{line } BA.$$

O5: If C and D are distinct points on the line AB, then A is on the line CD.

The next axiom puts us in two dimensions.

O6: If AB is a line, there is a point C not on this line.

Definitions. Points lying on the same line are said to be **collinear**. Three non-collinear points A, B, C determine a **triangle** ABC which consists of these three points, called **vertices**, together with the three segments BC, CA, AB, called **sides**.

The next axiom ensures that lines intersect in a well-behaved manner.

O7: If ABC is a triangle and if $[BCD]$ and $[CEA]$, then there is, on the line DE, a point F with $[AFB]$.

Aside: These axioms do seem very basic, and more or less state 'obvious' things about space around us. However, the axioms are already sufficient to prove some non-obvious things, e.g. Sylvester's Conjecture - If n points are not all collinear, there is at least one line containing exactly two of them (see 4.5).

Returning to routine things, we need more

Definitions. If A, B, C are three non-collinear points, the **plane** ABC is the set of all points collinear with pairs of points on one or two sides of the triangle ABC. A segment, interval, ray, or line is said to be **in** a plane if all its points are. An **angle** consists of a point O and two non-collinear rays going out from O. The point O is the **vertex** and the rays are the **sides** of the angle. If the sides are the rays OA and OB, or a_1 and b_1, the angle is denoted by $\angle AOB$ or $a_1 b_1$ (or $\angle BOA$, or $b_1 a_1$).

The penultimate order axiom is required to prevent us from creeping into three dimensions. (If we want more dimensions, we need new axioms analogous to **O6**.)

O8: All points are in one plane.

The final order axiom is very subtle and should remind you of the Dedekind cuts that appear in real analysis. If we chose to do so, we could use the axioms to define and describe the real field $\mathbb{R}$.

O9: For every partition of all the points on a line into two non-empty sets, such that no point of either lies between two points of the other, there is a point of one set which lies between every other point of that set and every point of the other set.

For our immediate purposes, this axiom is important in that it implies the existence of non-intersecting, and even *parallel*, rays.

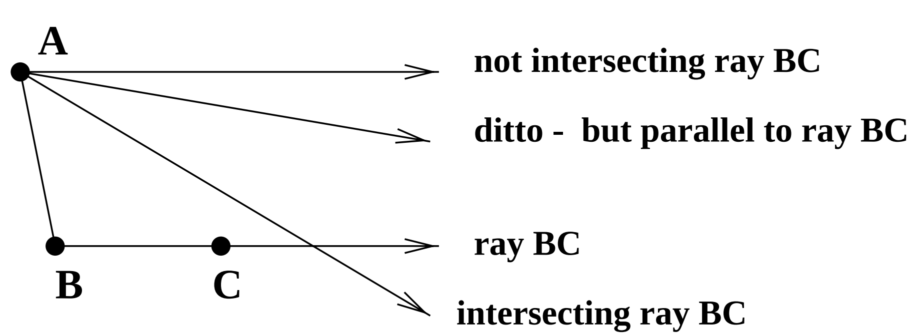


Figure 37: Parallel rays and continuity.

Congruence Axioms

The next set of axioms concerns a third primitive concept *congruence* (the others were *point* and *intermediacy* $[ABC]$). Thus congruence is undefined, but when we write $AB \equiv CD$, and say 'segment AB is congruent to segment CD', you should intuitively think that segment AB can be moved and placed exactly on top of segment CD. It is also correct to think of AB and CD as having the same length. But we have not and don't yet define 'length'.

C1: If A and B are distinct points, then on any ray going out from C there is just one point D such that $AB \equiv CD$.

C2: If $AB \equiv CD$ and $CD \equiv EF$, then $AB \equiv EF$.

C3:$AB \equiv BA$.

C4: If $[ABC]$ and $[A'B'C']$ and $AB \equiv A'B'$ and $BC \equiv B'C'$, then $AC \equiv A'C'$.

C5: If ABC and $A'B'C'$ are two triangles with $BC \equiv B'C'$, $CA \equiv C'A'$, $AB \equiv A'B'$, while D and D' are two further points such that $[BCD]$ and $[B'C'D']$ and $BD \equiv B'D'$, then $AD \equiv A'D'$.

We can now define *circles*, and for instance *right angles* (a right angle by definition is congruent to its supplement).

Absolute Geometry concerns those theorems that follow only from the above order and congruence axioms (**O1 - O9**, **C1 - C5**). Such theorems do not depend on an explicit axiom concerning parallels.

Typical absolute theorems (with Euclid's numbering in brackets) are

(a) the basic congruence theorems for triangles and angles:

(s.a.s.) (I-4), (a.s.a.) (I-26), (s.s.s.) (I-8), (v.o.a.) (I-15, equality of vertically opposite angles), P.A. (I-5, I-6: two sides in a triangle are equal if-f the opposite angles are equal)

(b) some basic constructions: angle bisectors (I-9), perpendiculars to lines (I-11, I-12)

(c) triangle inequality (I-20), exterior angle inequality (I-16)

(d) the existence of parallels because of equal alternate angles (I-27): see Theorem 4.8

These are rather dry theorems. But in fact, there are many less obvious and even surprising absolute theorems. We refer to [17], for example.

Now there are just two types of absolute geometry - the Euclidean geometry so familiar to us, and a very unfamilar type of non-Euclidean geometry called *hyperbolic* geometry (or *Lobachevskian* geometry).

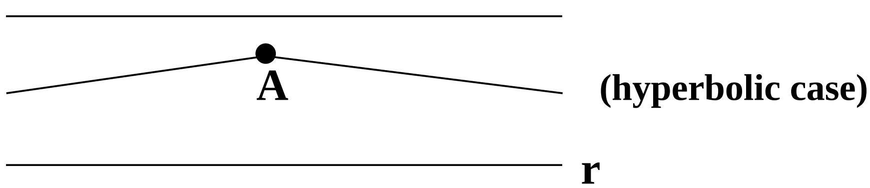
Just which geometry we happen to land in hinges upon whether the converse to (d) above is *true* (Euclid's theorem I-29) or *false* (hyperbolic geometry).

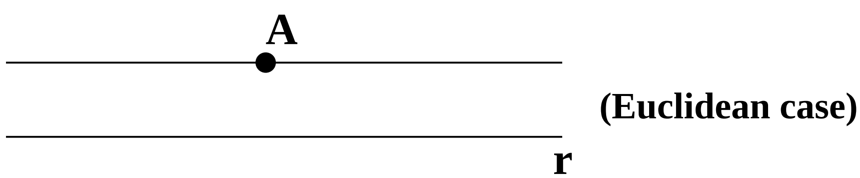
The most efficient way to distinguish the two geometries to choose <u>one</u> of the following axioms concerning parallels:

<u>PAR</u>:

THE EUCLIDEAN AXIOM. For some point A and some line r, not through A, there is not more than one line through A, in the plane Ar, not meeting r.

THE HYPERBOLIC AXIOM. For some point A and some line r, not through A, there is more than one line through A, in the plane Ar, not meeting r.

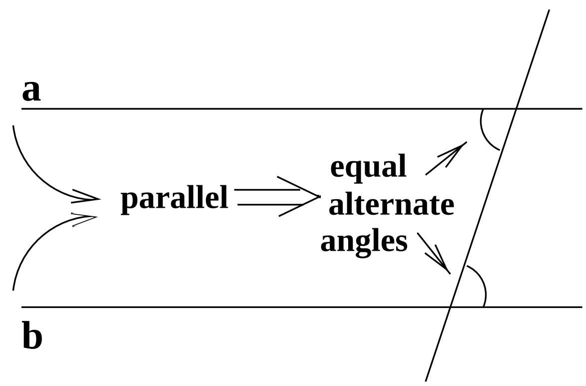




(The remaining axioms imply that *all* point-line pairs behave in the same way.)

Let's briefly recall from earlier sections what the Euclidean axiom does for our geometry. In Euclidean geometry (and **only** there), we define two lines a, b to be **parallel** if $a = b$ or a does not intersect b. Then we can prove the following converse to (d) (I-29):

If parallel lines a, b are cut by a transversal m, then the alternate angles are equal (Theorem 5.2):



Only because of this can we prove most of the familiar properties of the Euclidean plane: the angles in a triangle sum to $180°$, triangles with equal corresponding angles have sides in proportion, all of ordinary trigonometry, Pythagoras' theorem, the existence and use of Cartesian coordinates.

In hyperbolic geometry, this portion of the story unfolds very differently.