22 Frieze Patterns

A *frieze pattern* is any pattern which repeats at regular intervals along the whole length of some line: see Figure 76. Such patterns are commonly found on the lintels of ancient temples, and of course on more modern structures.

Each frieze pattern thus has a translation symmetry t; indeed it is symmetric under any power t^n of this translation ($n = 0, \pm1, \pm2, \ldots$). From another point of view, we may reconstitute the whole pattern by applying these translations to some basic decorative unit, called a *motif*.

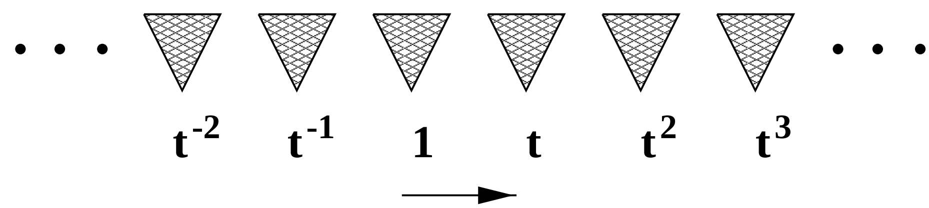


Figure 76: A typical frieze pattern.

22.1 Possible Symmetries

The symmetry group G of any frieze pattern always includes all powers of a *basic translation* t, which is one of two opposite and shortest non-trivial translations. (The other is t^{-1}.) Thus G has infinite order.

There may or may not be other sorts of symmetries. Here are the various possibilities:

(a) *Rotations.* In order that the pattern line be preserved, we can have only the identity or half-turn rotations. If there are indeed half-turns in G, then their centres must lie in a line a, which we call the *natural axis* for the pattern.

(b) *Reflections.* There could likewise be just the one reflection in some *natural axis a* parallel to the vector of t; or there might be reflections in mirrors perpendicular to the direction of translation; or there can be reflections of both these types.

(c) *Glides.* If there is a glide, it too must have the *natural axis a*.

22.2 A Flowchart for Frieze Patterns

There are only seven mathematically distinct types of frieze pattern. Any particular example can be classified by following the appropriate path **as far as possible** through the diagram below.

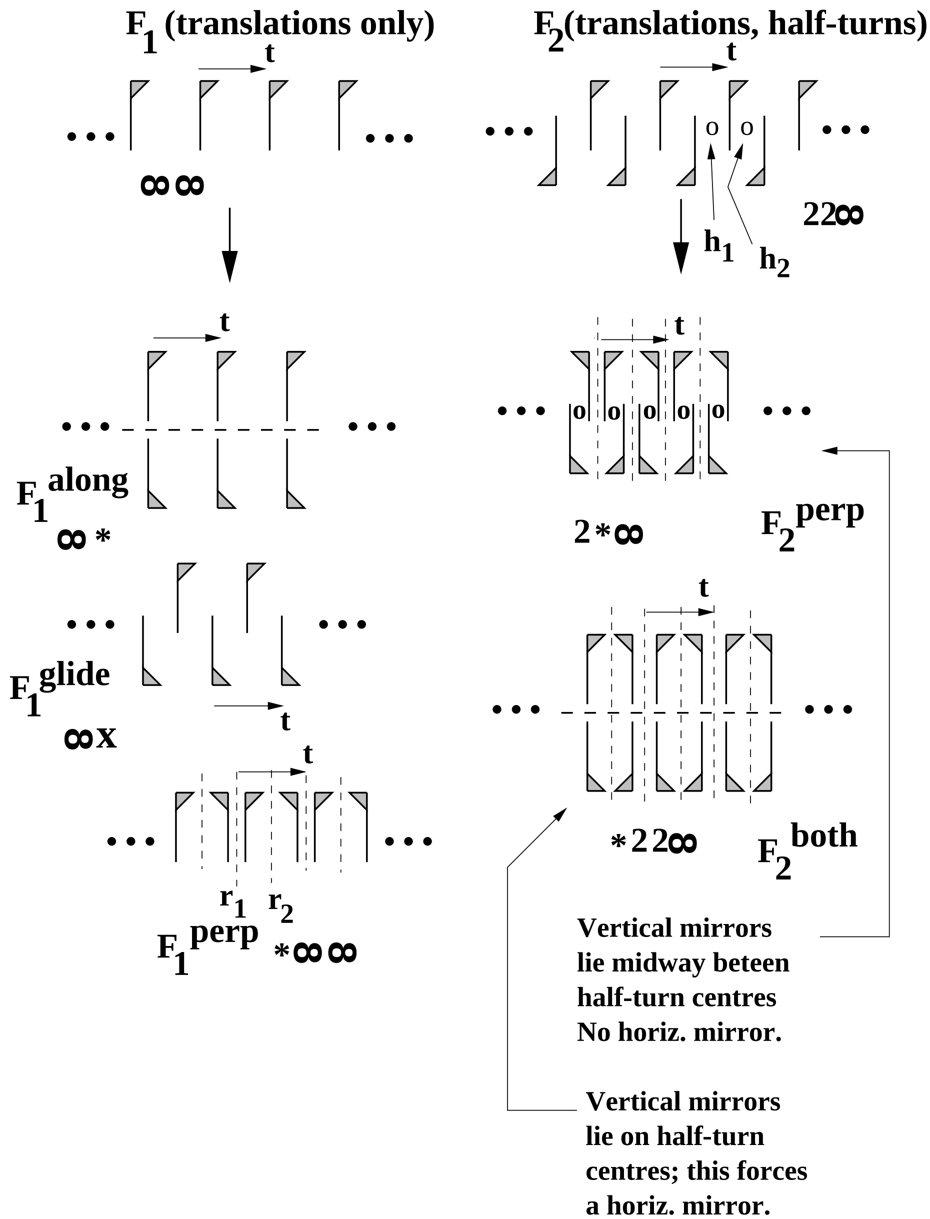


Figure 77: Flowchart for frieze patterns.

22.3 Notes

The **F** notation used here is my own, crude invention. In each, case you will also find J. H. Conway's very elegant **orbifold** notation for the frieze group (see [2]). Essentially, when we say that two frieze patterns are of the *same type*, we mean that their groups are 'identical'.

The first two types have only direct symmetries, i.e. translations and (perhaps) rotations.

$\mathbf{F_1}$ (or $\infty\infty$) : symmetric by no rotation except 1; no natural axis.

$\mathbf{F_2}$ (or 22∞) : symmetric by half-turns whose centres are equally spaced along the natural axis. Note that $t = h_1 h_2$.

The remaining five types admit as well opposite symmetries, i.e. reflections or glides or both.

$\mathbf{F_1^{\text{along}}}$ (or $\infty *$): the only reflection is in the natural axis a.

$\mathbf{F_1^{\text{perp}}}$ (or $*\infty\infty$): the only reflections are in equally spaced mirrors perpendicular to the translation vector; no natural axis. Note that $t = r_1 r_2$.

*** An interesting three-dimensional version of this pattern and group can be found on page 39 of the *New Yorker*, Feb. 23, 1957. Look at the copy in Figure 78 below.

$\mathbf{F_1^{\text{glide}}}$ (or $\infty\times$): glide but not reflection symmetries. There is a shortest glide g such that $g^2 = t$, where t is the shortest translation. This is the pattern formed by a set of footprints. There is a natural glide axis.

$\mathbf{F_2^{\text{both}}}$ (or $*22\infty$): reflection in the natural axis as well as reflections in lines perpendicular to the axis at the half-turn centres.

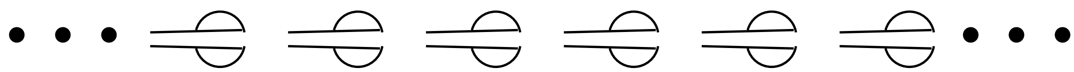
$\mathbf{F_2^{\text{perp}}}$ (or $2*\infty$): reflections only in lines perpendicular to the natural axis but midway between the half-turn centres. To make this pattern in the snow, walk in a straight line, turn around, then retrace your path, now taking care to place your feet exactly midway between the previous set of prints.

Here is the cartoon from the *New Yorker* mentioned above:

Figure 78: A bizarre frieze recreated in space.

22.4 Some Problems on Frieze Patterns

1. Identify from our class list the type of frieze:

(a) 

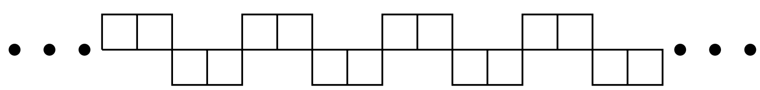
(b) 

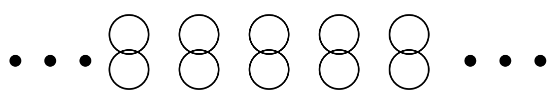
Also, show that patterns (a), (b) are mathematically distinct by exhibiting a type of symmetry possessed by one but not the other.

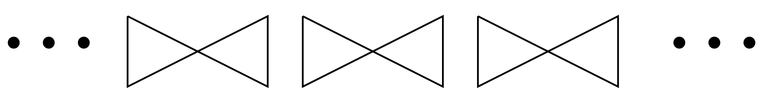
2. Sketch the frieze pattern created by repeating each of the following motifs at regular intervals along a horizontal straight line. Classify each pattern.

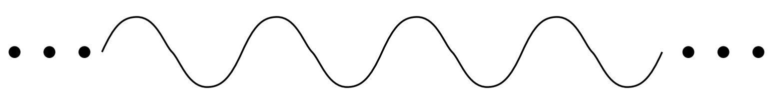
   (a) $\ni$ (b) Σ (c) H
   (d) L (e) M (f) N

3. The following patterns are of two types. Determine these and describe a symmetry for one of these patterns which is not a symmetry for another of the patterns.

(a) 

(b) 

(c) 

(d) 

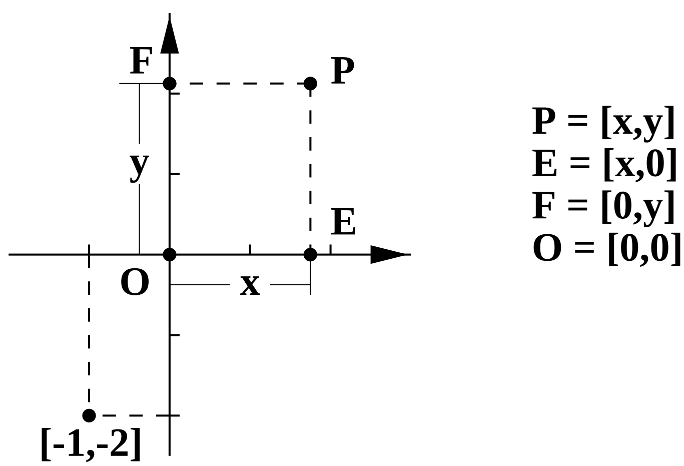
4. Look again at the barbershop in Figure 78. We have essentially a three dimensional version of the frieze pattern F_1^{perp}. The customer is facing one mirror, say for the reflection r_1. Presumably there is on the opposite wall, but not in the picture, another mirror for reflection r_2. What product of reflections takes the customer to the demon?

23 An Excursion into Coordinates and Trigonometry

By representing points using numbers we can convert geometric problems into algebraic problems (which are only sometimes easier in this new form). Also, similar techniques allow us to do problems in three or four or more dimensions, without having to visualize the objects concerned.

23.1 Coordinates

We first select perpendicular lines called the x-axis and the y-axis which cross at a point O called the *origin*. Next establish equal units on each axis. Now any point P has coordinates $[x, y]$, where in the *rectangle* $OEPF$, we take x to be negative if P lies to the left of the y-axis and take y to be negative if P lies below the x-axis.



Remarks:

(a) The signed numbers x and y represent real measurements; thus they are called *real numbers*. In order to determine these coordinates, we have to choose a scale of measurement on each axis. This is done by taking any convenient point on each axis as lying 1 unit from the origin. We need not (but often do) employ the same scale on each axis.

(b) We need not chose perpendicular $x-$ and $y-$axes. If these axes are not perpendicular, then we similarly obtain *oblique coordinates* $[x, y]$ (see Section 21.4). The rectangle $OEPF$ is then replaced by a *parallelogram* with sides parallel to the axes.

23.2 Trigonometric Functions

Draw a unit circle centred at the origin O. Next let $P = [x, y]$ be any point on this circle and let α be the angle from OA (the positive x-axis) to OP. Recall that α is taken to be positive if measured in the anti-clockwise sense, negative in the clockwise sense.

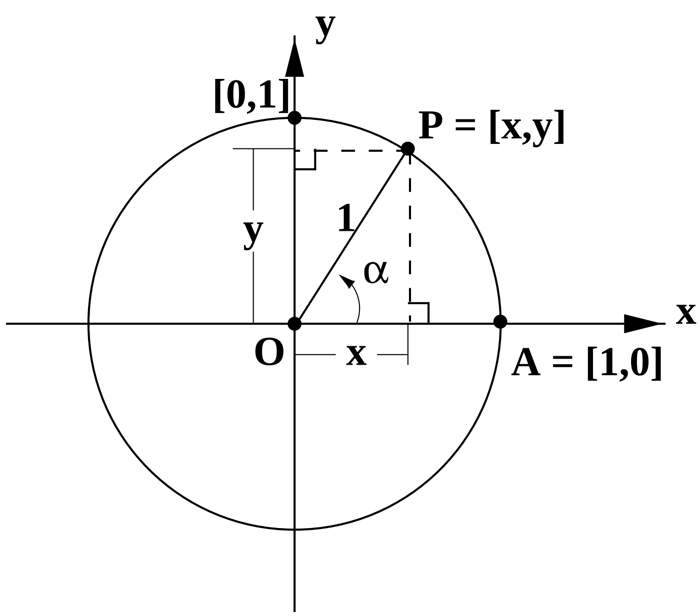


Figure 79: The unit circle.

Here are some definitions for the standard trigonometric functions: For any real number α

(i) $\cos\alpha = x$ and $\sin\alpha = y$.

(ii) $\tan\alpha = y/x$ and $\sec\alpha = 1/x$, if $x \neq 0$. Thus, α cannot be an odd multiple of $90°$. Note that $\tan\alpha = (\sin\alpha)/(\cos\alpha)$ and $\sec\alpha = 1/(\cos\alpha)$.

(iii) $\cot\alpha = x/y$ and $\csc\alpha = 1/y$, if $y \neq 0$. Here α cannot be any integer multiple of $180°$. Note that $\cot\alpha = (\cos\alpha)/(\sin\alpha)$ and $\csc\alpha = 1/(\sin\alpha)$.

We can make a few immediate conclusions:

(a) $\cos 0° = 1 = \sin 90°$.

(b) $\sin 0° = 0 = \cos 90°$.

(c) By Pythagoras (Theorem 8.2), the unit circle has equation

$$x^2 + y^2 = 1 \quad .$$

Hence, for all real α

$$\cos^2\alpha + \sin^2\alpha = 1 \quad .$$

23.3 Linear Equations

Draw through the origin the line l making an angle α with the positive x-axis. The constant

$$m = \sin\alpha / \cos\alpha = \tan\alpha$$

is called the *slope* of the line l. For any point $Q = [x, y]$ on the line, $\triangle OMQ \sim \triangle OLP$ (see Figure 80 below).

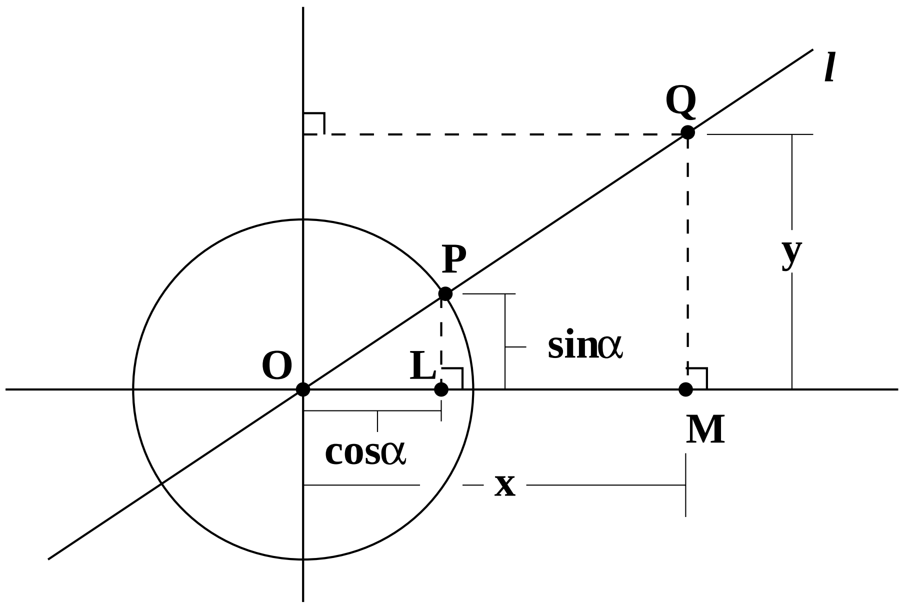


Figure 80: The geometry underlying a simple linear equation.

Thus $MQ/OM = LP/OL$, so that $y/x = \tan\alpha = m$. Hence the line l is described by the *linear equation*

$$y = mx \quad .$$

When $\alpha = 90°$, $\tan\alpha$ is undefined; nevertheless, l is still described by the linear equation $x = 0$.

23.4 Rotations

(a) Let r_α be the rotation through α about the origin O. Thus r_α is a transformation

$$r_\alpha : P \to P'$$

mapping $P = [x, y]$ to $P' = [x', y']$, say. This rotation is completely specified by requiring that $OP = OP'$ and $\angle POP' = \alpha$.

(b) To understand the action of r_α, let $Q' = [x', 0]$ be the point on both the x-axis and the vertical line through P'. Now rotate $\triangle Q'OP'$ through $-\alpha$ (i.e. 'backwards') to get the congruent $\triangle QOP$. Since r_α maps $\triangle QOP$ to $\triangle Q'OP'$, we see that

$$x' = OQ' = OQ \text{ and } \angle QOQ' = \alpha = \angle POP' \ .$$

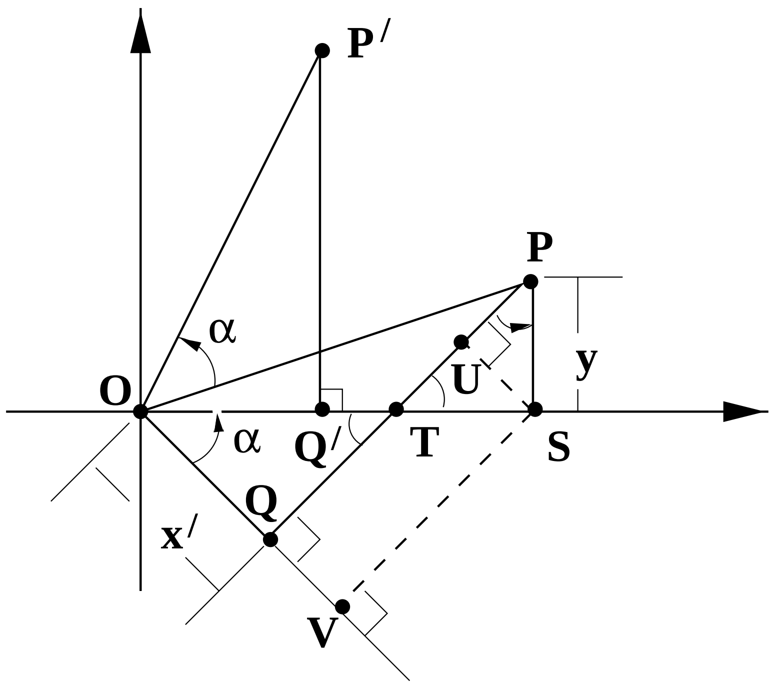


Figure 81: Rotations via coordinates.

Let line PQ meet the x-axis at T and draw SU perpendicular to PQ, where S is the point of intersection of the x-axis and the vertical line through P. Finally, draw SV perpendicular to line OQ. Note that $\angle UPS = \alpha$, and $US = QV$. Thus,

$$\begin{aligned} x' &= OQ \\ &= OV - QV \\ &= OV - US \\ &= (OV/OS)\,OS - (US/PS)\,PS \\ &= (\cos\alpha)x - (\sin\alpha)y \ . \end{aligned}$$

We can similarly compute y'. Thus the effect of r_α on $P = [x, y]$ is described by the following equations:

The rotation r_α in coordinate form

$$x' = x(\cos\alpha) - y(\sin\alpha) ,$$
$$y' = x(\sin\alpha) + y(\cos\alpha) .$$

Remark: Our proof is not completely general, since when α lies in other than the first quadrant, certain segment lengths must be subtracted rather than added (or vice versa). Mind you, these cases only require a slight modification of the diagram, and the equations above remain valid in all cases.

(c) *Example.* Let us describe a $60°$ rotation r about the origin. First note that an equilateral triangle with side length 2 has altitude $\sqrt{3}$. Thus $\cos 60° = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$. Hence,

$$x' = \frac{1}{2}x - \frac{\sqrt{3}}{2}y \text{ and } y' = \frac{\sqrt{3}}{2}x + \frac{1}{2}y .$$

As an application, find the vertices of the *regular hexagon* centred at the origin, one of whose vertices is $[1, 1]$ (see Section 25).

23.5 Trigonometric Identities

One very useful application of the rotation equations derived above is an easy way to produce the addition formulas for the sine and cosine functions.

We simply note that $r_\alpha r_\beta = r_{\alpha+\beta}$ (by Theorem 20.1) then apply these equal transformations to the point $[1, 0]$. Thus,

$$r_\alpha : [1, 0] \to [\cos\alpha, \sin\alpha]$$

so that

$$r_\alpha r_\beta : [1, 0] \to [\cos\alpha\cos\beta - \sin\alpha\sin\beta, \cos\alpha\sin\beta + \sin\alpha\cos\beta] .$$

On the other hand,

$$r_{\alpha+\beta} : [1, 0] \to [\cos(\alpha+\beta) , \sin(\alpha+\beta)] .$$

Comparing x and y coordinates, we obtain

The Addition of Angles Formulae.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta ,$$
$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta .$$

Most useful trigonometric identities can be derived from these two. For example, we get at once

The Double Angle Formulae.

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha ,$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha .$$

23.6 Reflections

(a) *Reflection p in the x-axis.*
Note that $p : P \to P'$, where P and P' have the same x-coordinate, but opposite y-coordinates. Thus,

$$p : [x, y] \to [x, -y] \ .$$

(b) *Reflection q in the line l given by $y = mx$.*
Recall that the slope $m = \tan\alpha$, where α is the angle at which the line l is inclined to the x-axis. Now $1 = \cos^2\alpha + \sin^2\alpha$, so that

$$1/(\cos^2\alpha) = 1 + \tan^2\alpha = 1 + m^2 \ .$$

Thus $\cos^2\alpha = 1/(1 + m^2)$, so that

$$\cos 2\alpha = \frac{2}{1+m^2} - 1 = \frac{1-m^2}{1+m^2} \ ,$$

and

$$\sin 2\alpha = \frac{2m}{1+m^2} \ .$$

(See the double angle identities in the previous section.) But $r_{2\alpha} = pq$ is a rotation through 2α about the origin, so

$$q = p^{-1} r_{2\alpha} = p r_{2\alpha}$$

Thus

$$q : [x, y] \xrightarrow{p} [x, -y] \xrightarrow{r_{2\alpha}} [x(\cos 2\alpha) + y(\sin 2\alpha), x(\sin 2\alpha) - y(\cos 2\alpha)] \ .$$

Conclusion. The reflection in the line through the origin with slope m maps $[x, y]$ to $[x', y']$, as described in the following equations:

Reflection in the line $y = mx$.

$$x' = \{x(1 - m^2) + y(2m)\}/(1 + m^2) \ ,$$

$$y' = \{x(2m) - y(1 - m^2)\}/(1 + m^2) \ .$$

(c) For example, reflection in the line $x + y = 0$ (or $y = (-1)x$) maps $[x, y]$ to $[-y, -x]$.

23.7 Translations

A translation t is specified by describing its effect on any convenient point, say the origin.

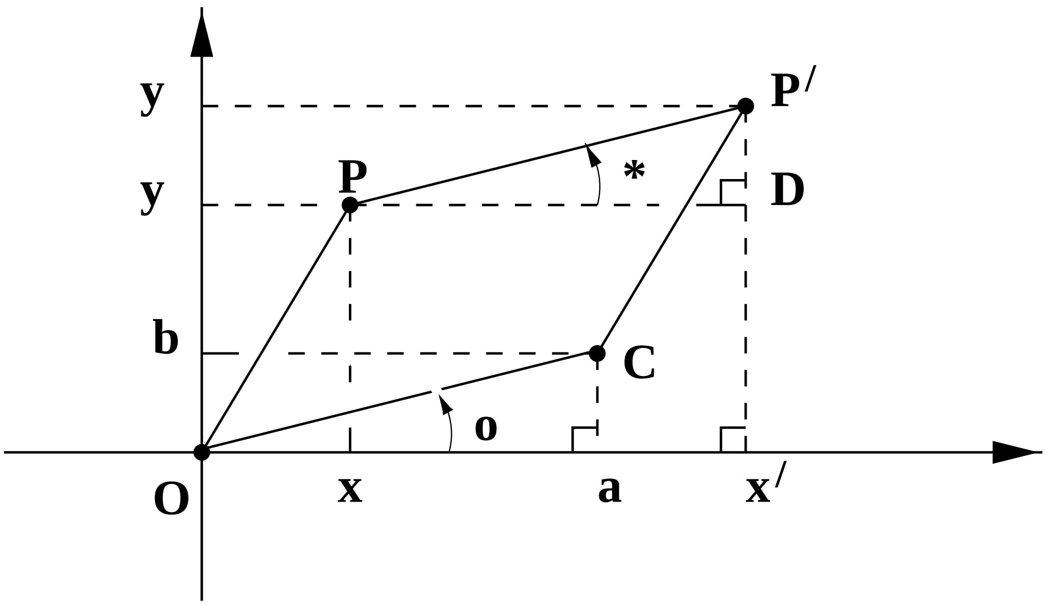


Figure 82: Translations in coordinate form.

First of all, suppose

$$t : O = [0, 0] \to C = [a, b] \text{ and } t : P = [x, y] \to P' = [x', y']$$

as shown in Figure 82. Since $OCP'P$ is a parallelogram, we have $\circ = *$ and $OC = PP'$. Thus by (a.s.a.),

$$\triangle OAC \equiv \triangle PDP' \ .$$

Hence,

$$x' - x = PD = OA = a \ ,$$

so $x' = x + a$, and similarly $y' = y + b$.

Conclusion: If a translation t maps the origin $O = [0, 0]$ to $[a, b]$, then t maps $[x, y]$ to $[x', y']$ as described in the following equations:

The translation t in coordinate form

$$t : [x, y] \to [x + a, y + b]$$

23.8 Matrix Operations

(a) A *matrix* P is any rectangular array of numbers. If there are m rows and n columns in this array, we say that P has *size* or *dimensions* $m \times n$. The number in row i and column j is called the ij^{th} *entry*.

Two matrices are quite naturally said to be *equal* when:

(i) they have the same size, and

(ii) entries in *all corresponding positions* are equal.

We shall also see that the sensible way to add two matrices is to add corresponding entries. Here is a formal

Definition 14 *The* sum *of two $m \times n$ matrices P and C is the $m \times n$ matrix $P + C$ whose ij^{th} entry is the sum of the ij^{th} entries in P and in C.*

Matrix addition obeys various familiar algebraic laws. For example, if P, C and D are matrices of the same size, then

$$(P + C) + D = P + (C + D) \quad ,$$

(addition of matrices is *associative*).

(b) We are already familiar with certain matrices. Each point in the plane is represented by a 1×2 matrix

$$P = [x, y]$$

In this case, and elsewhere in these notes, the entries x and y are real numbers. Referring back to Section 21.4, we see that it also makes sense to call P a *row vector*.

(c) *Translations and Matrix Addition.*

(i) Any translation t or u is determined by its effect on the origin O. Suppose t has vector $\vec{OC}$, where $C = [a, b]$. We observed in section 23.7 above that

$$t : [x, y] \to [x + a, y + b]$$

If we translate this into matrix notation, we conclude that the translation t is described by

$$P' = P + C$$

Since the translation vector $\vec{OC}$ is a position vector (i.e. has the origin as initial point), it is convenient to identify $\vec{OC}$ with the matrix $[a, b]$.

(ii) Example. If $t : O \to O = C$, then $t = 1$ is the identity. In this case,

$$P' = P = P + O \quad ,$$

for all points $P = [x, y]$. In other words, the *zero matrix* $O = [0, 0]$ serves two purposes. It represents the origin as a point, and it acts as an identity for matrix addition:

$$[x, y] = [x, y] + [0, 0] \quad .$$

(iii) Suppose that another translation $u : O \to D = [\hat{a}, \hat{b}]$, so that

$$u : Q \to Q + D, \text{ for all points } Q.$$

Thus,

$$P \xrightarrow{t} (P + C) \xrightarrow{u} (P + C) + D = P + (C + D) \ .$$

In terms of coordinates, we have:

$$[x, y] \xrightarrow{t} [x + a, y + b] \xrightarrow{u} [(x + a) + \hat{a}, (y + b) + \hat{b}]$$

$$= [x + (a + \hat{a}), y + (b + \hat{b})] \ .$$

Thus, we have verified from another point of view that if t is a translation with vector described by the matrix C, and u is a translation with vector D, then tu is a translation (with vector $C + D$).

Compare this with Figure 82 in Section 23.7 above. We conclude that the fourth vertex of a parallelogram with vertices C, O and D is $C + D$.

(iv) In the previous example, we could in particular let $u = t^{-1}$, so that

$$1 = tt^{-1} = tu \ .$$

Here then the zero vector $O = C + D$, so that

$$[0, 0] = [a, b] + [\hat{a}, \hat{b}] = [a + \hat{a}, b + \hat{b}] \ .$$

Hence, $D = [\hat{a}, \hat{b}] = [-a, -b]$, which we sensibly call $-C$.

Conclusion: If t has vector $C = [a, b]$, then t^{-1} has vector $-C = [-a, -b]$.

(d) *Matrix Multiplication.* The product of two matrices P and R is defined in a most unexpected way. Although matrix multiplication may at first seem peculiar, it has many applications and is tailor-made to represent rotations.

(i) **Definition 15** [14] *Suppose P is a $k \times m$ matrix and R is an $m \times n$ matrix. Then the* product *PR is a $k \times n$ matrix. To calculate the entry in row i and column j of PR:*

- *simultaneously scan across row i of P and down column j of R.*
- *multiply corresponding entries, and*
- *add these m products to get the ij^{th} entry in PR.*

[14] It is important to note that the matrix product PR is defined only when the dimensions match as shown:

$$\#\text{columns of } P = m = \#\text{rows of } R \ .$$

(ii) Example.

$$P = \begin{bmatrix} 1 & 7 & -1 \\ 2 & 8 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$(2 \times 3) \quad \text{times} \quad (3 \times 1)$$

Thus the inner dimensions match and we have

$$PR = \begin{bmatrix} 33 \\ 48 \end{bmatrix} \qquad \begin{array}{l} 33 = 1 \cdot 4 + 7 \cdot 5 + (-1) \cdot 6 \\ 48 = 2 \cdot 4 + 8 \cdot 5 + 0 \cdot 6 \end{array}$$

with size (2×1). Note, however, that that the product RP is not even defined.

(e) *The rotation r_α through α about the origin.*
Let $P = [x, y]$ be the 1×2 matrix describing an arbitrary point in the plane. Taking a hint from Section 23.4, we construct the following 2×2 *rotation matrix* for r_α :

$$R = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Then

$$PR = \begin{bmatrix} x, y \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$
$$= [x(\cos\alpha) - y(\sin\alpha), x(\sin\alpha) + y(\cos\alpha)] \ .$$

If you look back at the equations describing r_α in section 23.4, you will see that $PR = P'$.

Conclusion: The rotation r_α is given in matrix form by the matrix product

$$P' = PR \ ,$$

where P is any point and R is the rotation matrix described above.

(f) Example. Let $\alpha = 0°$. The corresponding rotation $r_0 = 1$ (the identity). Thus, we naturally call the corresponding rotation matrix

$$\begin{bmatrix} \cos 0° & \sin 0° \\ -\sin 0° & \cos 0° \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the *identity matrix* I. Indeed, for all $P = [x, y]$

$$PI = P.$$

(g) *Matrix Algebra* . The algebra of matrices is not totally like the ordinary algebra of real numbers. For example, there is sometimes no sensible way of saying that a non-zero matrix has a 'reciprocal'. Furthermore, matrix multiplication is not always commutative. Nevertheless, to a limited extent, matrices do interact algebraically as one would expect.

To illustrate this, let us describe the rotation s through α about a centre $C = [a, b]$. We already know that the rotation r_α through α about the origin O is represented in matrix form by

$$P \to P' = PR \ ,$$

where R is the rotation matrix described above. Next let t be the translation with vector $\vec{OC}$.

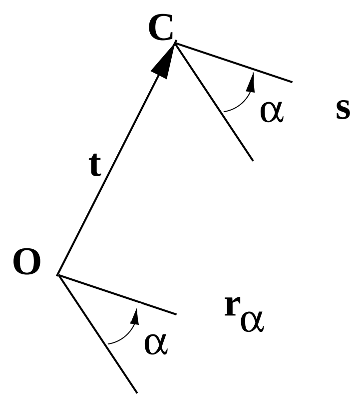


Figure 83: A rotation s conjugate to r_α.

Now $t^{-1} r_\alpha t$ is a rotation with angle $(-0°) + \alpha + (0°) = \alpha$, and which maps

$$C \to O \to O \to C \ .$$

Hence, $s = t^{-1} r_\alpha t$.

Thus we can easily desribe s by matrices:

$$P \overset{t^{-1}}{\to} (P - C) \overset{r_\alpha}{\to} (P - C)R \overset{t}{\to} (P - C)R + C \ .$$

Conclusion: The rotation s with centre C and angle α is represented in matrix form by

$$\begin{aligned} s : P \to P' &= (P - C)R + C \\ &= PR + (C - CR) \end{aligned}$$

For example, let's describe the 30° rotation with centre $C = [4, 1]$. This is given by

$$[x', y'] = [x - 4, y - 1] \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} + [4, 1] \ .$$

Thus,

$$\begin{aligned} x' &= \frac{\sqrt{3}}{2}x - \frac{y}{2} + \frac{9}{2} - 2\sqrt{3} \\ y' &= \frac{x}{2} + \frac{\sqrt{3}}{2}y - 1 - \frac{\sqrt{3}}{2}. \end{aligned}$$

23.9 Distance and Angle Measurement

Much of our day-to-day work in geometry concerns the measurement of distances and angles. We have a good understanding of how these things behave under motions. But how do we make measurements with coordinates?

(a) **Distance**. It is not at all difficult to measure the distance $d = d(P, Q)$ between two points $P = [x_1, y_1]$ and $Q = [x_2, y_2]$, as in Figure 84.

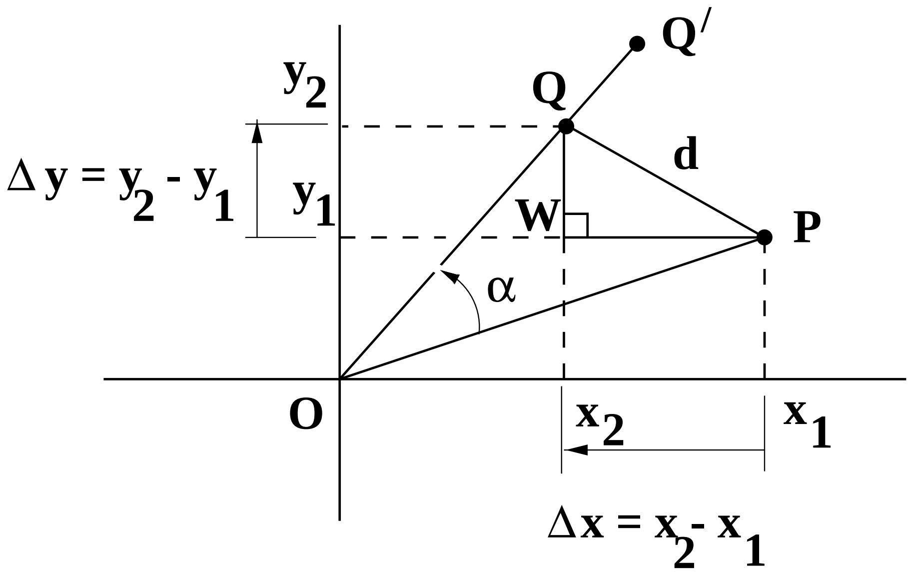


Figure 84: Distance and angles from coordinates.

Observe that the vertical line through Q meets the horizontal line through P in the point $W = [x_2, y_1]$, thus forming a right triangle $\triangle PWQ$. Note that the hypotenuse has length d, whereas the remaining sides have *signed* lengths $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$. It would be awkward to keep track of these signs, but this is unnecessary anyway, since in Pythagoras' theorem (8.2) all distances are squared:

$$d^2 = (\Delta x)^2 + (\Delta y)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \ .$$

You can now compute d after finding the square root.

(b) **Angles**. This is a little trickier, though again we can extract the main ideas from Figure 84. We shall calculate $\alpha = \angle POQ$.

Let r_α be the rotation with centre O and angle α described in Section 23.4. Thus

$$r_\alpha : P \to Q' \ ,$$

where

$$Q' = [x_1 \cos\alpha - y_1 \sin\alpha \, , x_1 \sin\alpha + y_1 \cos\alpha]$$

lies on line OQ. (Most likely, though, $Q \neq Q'$.) Nevertheless, both Q and Q' produce with O the same slope for this line, so that

$$\frac{y_2}{x_2} = \frac{x_1 \sin\alpha + y_1 \cos\alpha}{x_1 \cos\alpha - y_1 \sin\alpha} \ .$$

Cross multiplying and collecting the coefficients of $\cos\alpha$ and $\sin\alpha$ we get

$$(x_1 y_2 - x_2 y_1)\cos\alpha = (x_1 x_2 + y_1 y_2)\sin\alpha$$

so that

$$A\cos\alpha = B\sin\alpha \ ,$$

where

$$A = x_1 y_2 - x_2 y_1 \ \text{ and } \ B = x_1 x_2 + y_1 y_2 \ .$$

Since $\sin^2\alpha = 1 - \cos^2\alpha$, squaring both sides of the equation yields

$$A^2 \cos^2\alpha = B^2(1 - \cos^2\alpha) \ .$$

Thus,

$$\cos^2\alpha = \frac{B^2}{A^2 + B^2} \ ,$$

where

$$\begin{aligned} A^2 + B^2 &= (x_1 y_2 - x_2 y_1)^2 + (x_1 x_2 + y_1 y_2)^2 \\ &= x_1^2 y_2^2 + x_2^2 y_1^2 + x_1^2 x_2^2 + y_1^2 y_2^2 \\ &= (x_1^2 + y_1^2)(x_2^2 + y_2^2) \ . \end{aligned}$$

Taking the square root, we finally obtain [15]

$$\cos\alpha = \frac{x_1 x_2 + y_1 y_2}{\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}} \ .$$

This important equation contains in three places an algebraic quantity that deserves a special name.

[15] Careful! There must be some doubt as to whether we take the positive or negative root. To see which, take the simplest case, in which $P = Q$, so that $\alpha = 0^\circ$. But then $x_1 = x_2$ and $y_1 = y_2$, so that

$$1 = \cos 0^\circ = \frac{\pm(x_1^2 + y_1^2)}{x_1^2 + y_1^2} = \frac{\pm 1}{1} \ .$$

We are forced to chose the positive root.

Definition 16 *The* dot product *of* $P = [x_1, y_1]$ *and* $Q = [x_2, y_2]$ *is the real number*

$$P \cdot Q = x_1 x_2 + y_1 y_2 \ .$$

Remark. The *transpose* of the 1×2 matrix Q is the 2×1 matrix Q^T obtained by writing its row as a column. Then we can think of the dot product of P and Q as the 1×1 matrix

$$P \cdot Q = PQ^T = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = [x_1 x_2 + y_1 y_2] \ .$$

Note furthermore that $P \cdot P = x_1^2 + y_1^2$. It follows then that the angle α is given by

$$\cos\alpha = \frac{P \cdot Q}{\sqrt{(P \cdot P)(Q \cdot Q)}} \ .$$

(c) *The Law of Cosines.* The triangle above was special in that one vertex was the origin O. Suppose, however, that we are presented with a general triangle $\triangle ABC$, with angles α , β and γ, and opposite sides a , b and c (see Figure 85).

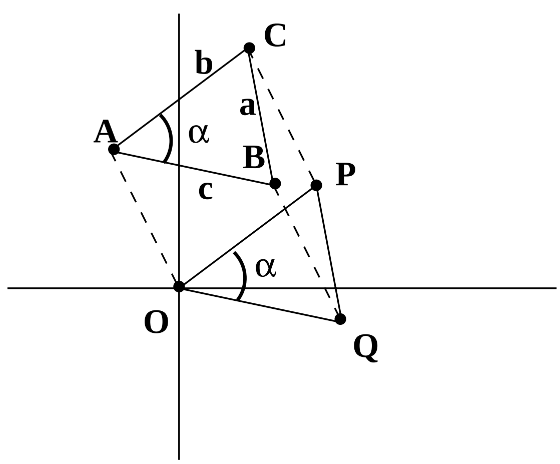


Figure 85: Some trigonometric calculations.

How do we calculate say α in this case? Well, imagine that we apply the translation t with vector $\vec{AO}$. Then α is unchanged but

$$\begin{aligned} A &\to A - A = O \ , \\ B &\to B - A = Q \text{ (say)} \ , \text{ and} \\ C &\to C - A = P \text{ (say)} \ . \end{aligned}$$

Notice that $b^2 = P \cdot P$, $c^2 = Q \cdot Q$ and

$$\begin{aligned} a^2 &= (P - Q) \cdot (P - Q) \\ &= P \cdot P + Q \cdot Q - 2P \cdot Q \\ &= b^2 + c^2 - 2P \cdot Q \ . \end{aligned}$$

On the other hand, we have from above that

$$\cos\alpha = \frac{P\cdot Q}{\sqrt{(P\cdot P)(Q\cdot Q)}} = \frac{b^2+c^2-a^2}{2bc} \quad .$$

This last result is often called **the law of cosines**.

(d) *Example* Let us suppose, for example, that $A = [-1, 2]$, $B = [3, 0]$ and $C = [2, 4]$. The distance formula immediately gives $a = \sqrt{17}$, $b = \sqrt{13}$ and $c = \sqrt{20}$.

Then the law of cosines gives

$$\cos\alpha = \frac{13+20-17}{2\sqrt{13}\sqrt{20}} = \frac{4}{\sqrt{65}} \simeq 0.4961 \quad .$$

Your calculator will give $\alpha \simeq 60.255°$. One can similarly calculate β and γ. Or we may use the law of sines from problem 5 on page 62.

Thus, $\sin\beta = (b\sin\alpha)/a \simeq 0.7592$, so that $\beta \simeq 49.399°$. Likewise, we get $\gamma \simeq 70.346°$. As a check, notice that

$$\alpha+\beta+\gamma \simeq 180.000° \ ,$$

despite all errors in calculation.

23.10 A Few Problems on Coordinates and Matrices

1. Represent each of the following isometries in matrix form, namely

$$[x, y] \to \text{some sum or product of matrices.}$$

   (a) The translation t taking $[1, 3]$ to $[-1, 5]$.
   (b) t^{-1}, for t in (a).
   (c) The translation taking $[1, 4]$ to $[1, 4]$.
   (d) The reflection r in the line of slope 4 through the origin O.
   (e) The rotation s through $60°$ about O. (Use exact values for the trig. functions.)
   (f) s^{-1}, for s in (e).
   (g) The identity 1 (considered as a rotation through $0°$).

2. (a) Give the 2×2 matrices A, B representing the rotations through $65°$ and $-23°$ (respectively) about O.
   (b) Compute AB to 3 place accuracy.
   (c) Compare AB with C, the matrix for the rotation through $42°$ about O. Explain.

3. (a) Represent in matrix form the translation $t : [0, 1] \to [3, 3]$. To which point Q does t send O?
   (b) Find the slope m of the line OQ.
   (c) Represent in matrix form the reflection r in line OQ.
   (d) Represent in matrix form the glide $g = tr$.

4. (a) Give the matrices which represent rotations about O through
      (i) $15°$: matrix A
      (ii) $35°$: matrix B
      (iii) $50°$: matrix C
   (b) Check that $AB = C$ (up 3 decimal place accuracy).
   (c) Accurately plot the points $[1, 0]$, $[1, 0]\,B$, $[2, 1]$ and $[2, 1]\,B$. Check by protractor that both $[1, 0]$ and $[2, 1]$ have been rotated by $35°$.