24 Geometrical Constructions

There are many useful instruments available for geometrical constructions, such as the ruler, compasses, T-square, protractor, set squares, etc. However, in a tradition beginning with the ancient Greeks and continuing into modern times, mathematicians usually allow only the first two instruments - the ruler (or straightedge) and compasses. These rather arbitrary restrictions are akin to the ‘arbitrary’ rules underlying any game, be it baseball or chess.

24.1 The Ruler

Our ruler is really a single straightedge, with no markings (like the smooth edge of a hacksaw blade). Thus one is not allowed to use centimeter or inch marks, nor is one allowed to simultaneously use both sides of an ordinary ruler.

(a) *What can a ruler do*? Basically all we are allowed to do with a ruler is position its edge over two distinct points A and B, then draw a portion of the line AB. In practice, this process is inexact; however, for the purposes of the mathematical theory describing this and other constructions, we assume that the line drawn is ‘perfectly thin’ and is exactly in its proper position.

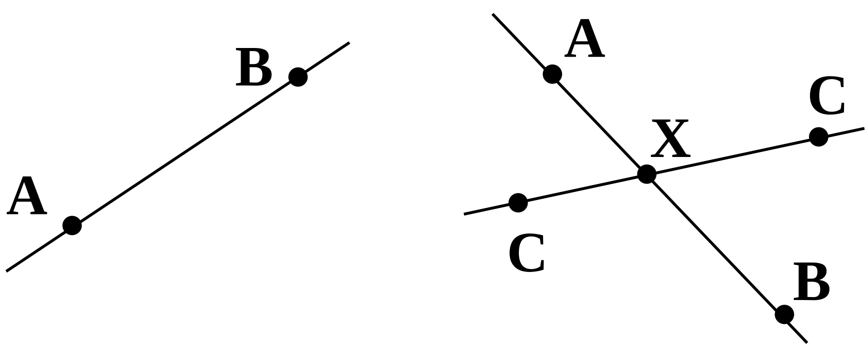


Figure 86: Constructions with the ruler.

(b) *What can we then find*? Having drawn lines AB and CD, we can pick out the point X (if any) where the two lines cross (see Figure 86 above).

(c) By the way, that part of geometry which essentially concerns only those constructions which can be performed using just a ruler i.e. straightedge, is called *projective geometry*. Here is an unusual projective construction, which is situated in the hinterland between Euclidean and projective geometry:

Suppose that M is the midpoint of segment AB and that P is a point not on the line AB. Using *only a ruler*, give an exact contruction for the line through P which is parallel to AB. (See Figure 87; the construction isn't obvious, and we don't give the answer here.)

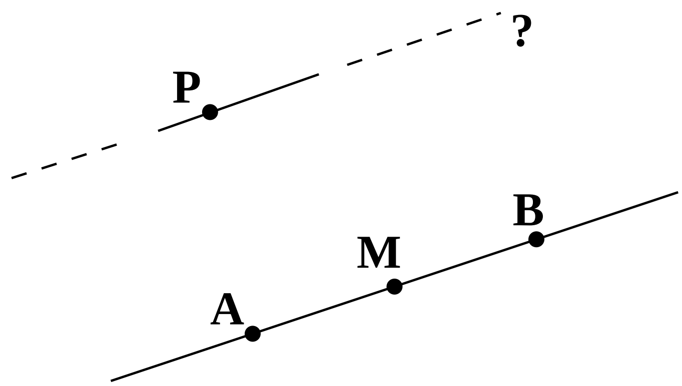
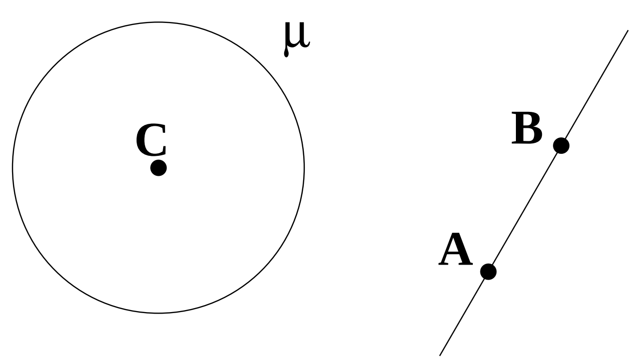


Figure 87: A projective construction.

24.2 Compasses

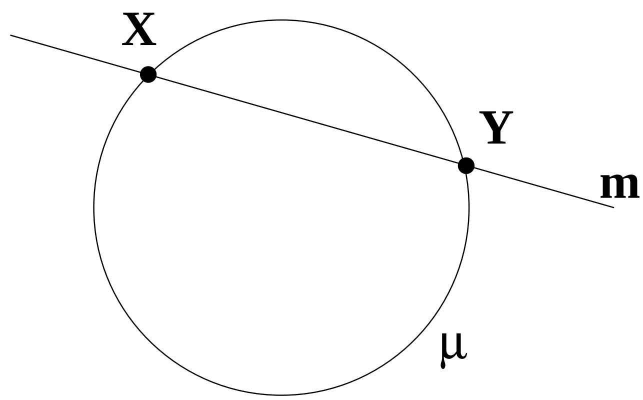
The word 'compasses' really refers to a single instrument. We assume that our compasses can be opened by an arbitrarily large or small amount. In practice, we need not push the instrument to such extremes, so this is no real restriction. Also, we assume that the attached pencil is capable of drawing 'perfectly thin' circular arcs.

(a) *What can compasses do*? They can be used to draw the circle μ with any centre C and radius AB, for two known points A and B.

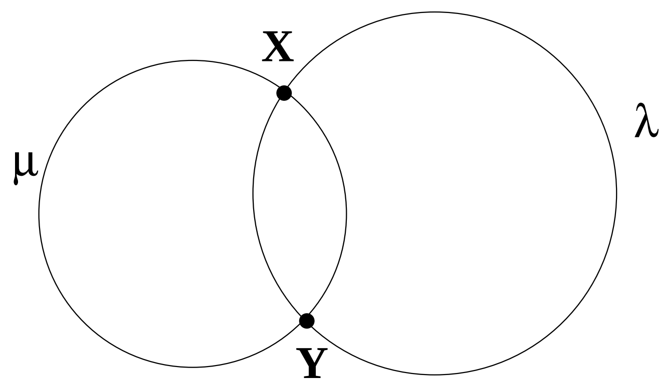


(b) *What can then be found*? If you think about it, we can find only:

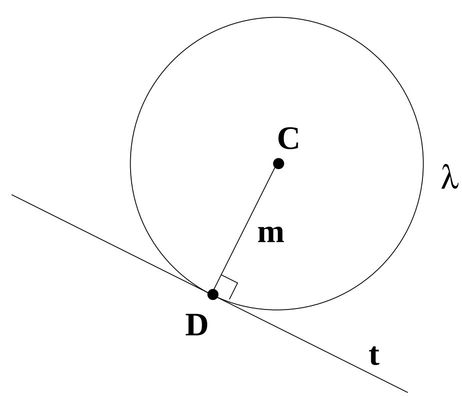
(i) the points X, Y where the circle μ crosses a line m,



(ii) or the points X, Y where two circles μ and λ cross.



(c) Note that the opening for the compasses can be set only by using two points. Keeping this in mind, suppose that you want to draw the circle λ with centre C, which is tangent to the line t.



Now according to the ideal rules of our construction game, one cannot simply increase the opening of the of the compasses until the circle centred at C just touches t.[16] Instead, we must first find the point D where the circle λ touches t. This is done by constructing the line m through C and perpendicular to t; then D is the point on lines m and t. Now we may set the radius to CD and draw the circle λ.

24.3 Basic Constructions

Review these basic constructions, or learn them if they are unfamiliar. Details can be found in the Elements, or any basic geometry book.

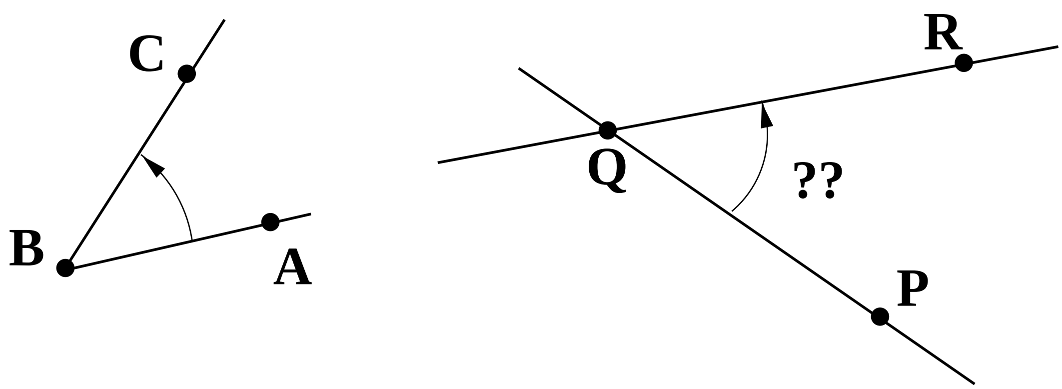
(a) Given a line m and a point P construct the line through P which is perpendicular to m when :

(i) P lies on m, or

(ii) P does not lie on m.

(b) Bisect a given angle.

(c) Duplicate a given angle. That is, given $\angle ABC$ and segment PQ draw $\angle PQR = \angle ABC$.



(d) Given a point P not on line a draw the line b through P which is parallel to a.

(e) Use parts (a) and (b) to construct a square, then a regular octagon, then a regular sixteen sided polygon.

[16]Of course, in practical work one might do just that. For the moment, however, let us rigidly follow the rules of the construction game.

(f) Fix a unit of measurement—that is, suppose that a chosen segment CD has length 1. Now given two segments of lengths a and b, say

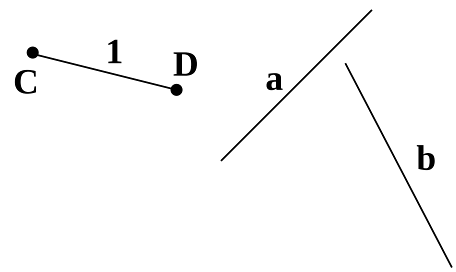


Figure 88: Arithmetic via geometric constructions.

construct segments of length:

(i) $a + b$,

(ii) $a - b$ (if $a > b$) ,

(iii) ab ,

(iv) a/b ,

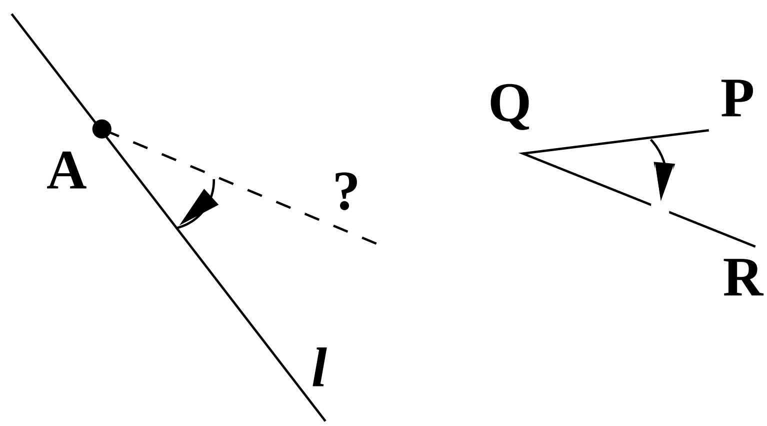
(v) $\sqrt{ab}$.

24.4 Some Basic Ruler & Compasses Constructions

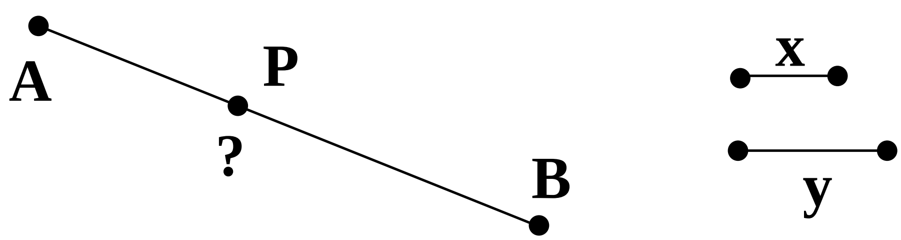
. For each R-C construction below describe, and neatly perform the construction; then prove that your construction actually works using results from the notes. (Need a hint? -consult any standard geometry text.)

1. Find the perpendicular to line ℓ
   (a) at a point A on ℓ;
   (b) from a point A not on ℓ.
2. Bisect a given angle $\angle ABC$.
3. Bisect a given segment AB.
4. Find the perpendicular bisector of segment AB.

5. From a point A on a line ℓ construct a line making a given $\angle PQR$ with ℓ.



6. Through a point A (not on line ℓ), draw the line m parallel to ℓ.
7. Give definitions for and construct the centroid, in-centre, in-circle, circum-centre and circum-circle for $\triangle ABC$.
8. Construct the circle through three given points.
9. Divide a given segment *internally* AB in the ratio $\dfrac{x}{y}$, where x and y are given lengths. That is, find P with $\dfrac{AP}{PB} = \dfrac{x}{y}$:



10. By searching a traditional geometry test, determine what it means to divide a given segment AB *externally* in the ratio $\frac{x}{y}$. Describe this construction.
11. Given a segment of unit length (i.e. length l), construct a segment of length $\sqrt{m}$, for any positive integer m. (Hint: see problem 3c on page 61.)
12. Construct angles of $90°, 60°, 45°, 30°$.
13. Which constructions above depend on (a) (s.a.s.) only? (b) both (s.a.s.) and (par. ax.)?
14. (Library Search: Is it possible to give an R-C construction which will trisect any given angle? Who first answered this question? What sort of person is known as an 'angle trisector'? (There are some everywhere - New Brunswick included.)

24.5 More Constructions.

Clearly describe how to perform each of the following ruler-compasses constructions and prove the correctness of your procedure. Some of these construcions require only the basic axioms (i.e. not parallelism). But despite that you may use any results in Euclidean geometry that seem helpful.

1. Draw a right-angled isosceles triangle given one of the equal sides.
2. Construct a right-angled triangle such that one of the sides of the right angle equals a given line segment (in length), whereas the hypotenuse equals twice the given line segment.
3. Describe a circle to pass through three given points not in a straight line.
4. Construct a triangle being given the base, one of the base angles and the difference between the other two sides.
5. Construct an isosceles triangle given the perimeter and the altitude.
6. Given a $\triangle ABC$ and a segment XY, draw a straight parallel to BC, terminated by AB and AC (produced if necessary), and equal in length to XY.
7. Suppose P is a point within angle $\angle ABC$. Draw a segment through P, terminated by AB and BC, and bisected at P.
8. Suppose D is a point on side KL of $\triangle KLM$. Find a point E on KM produced such that DE is bisected by LM.
9. P, Q, R are three points not in the same straight line. Construct a triangle such that P, Q, R are the midpoints of the sides.
10. From a point P on the base of a triangle the lines perpendicular to the other two sides are drawn. Locate P so that the difference of the lengths of these two lines equals a given length.
11. Describe a circle to pass through two given points and to have its centre on a given line.
12. Find a point equidistant from three given points not in the same straight line.
13. Find a point equidistant from two given points and at a given distance from a given line.
14. An arc of a circle being given, show how to complete the circle.
15. A is a given point within a given circle. Through A draw a chord that is bisected at A.
16. Construct a right-angled triangle having given the hypotenuse and one side.
17. Given a circle ABC, circumscribe about it an equilateral triangle.
18. Find a point P within the triangle $\triangle ABC$ such that $\angle APB = \angle BPC = \angle CPA$. When is the solution impossible?

25 Regular Polygons

25.1 Polygon

(a) A *polygon* $\mathcal{P}$ is the figure formed by n points

$$P_0\ ,\ P_1\ ,\ \ldots, P_{n-1}\ \ \text{(the \textit{vertices})}$$

joined consecutively by the n segments

$$P_0P_1\ , \ldots, P_{n-2}P_{n-1}, P_{n-1}P_0\ \ \text{(the \textit{edges})}.$$

It is convenient here to read subscripts mod$n$.

(b) A polygon with n vertices may be called an *n-gon*. If all these vertices lie in one plane, then the polygon is said to be *planar*. Otherwise, the polygon is *skew*.

(c) A *convex polygon* is one which encloses a plane convex region. Recall that (by definition) a *convex region* must contain the full line segment joining any two of its points. (Intuitively, a convex region has no hidden corners: *any* two people in a convex room can see one another.)

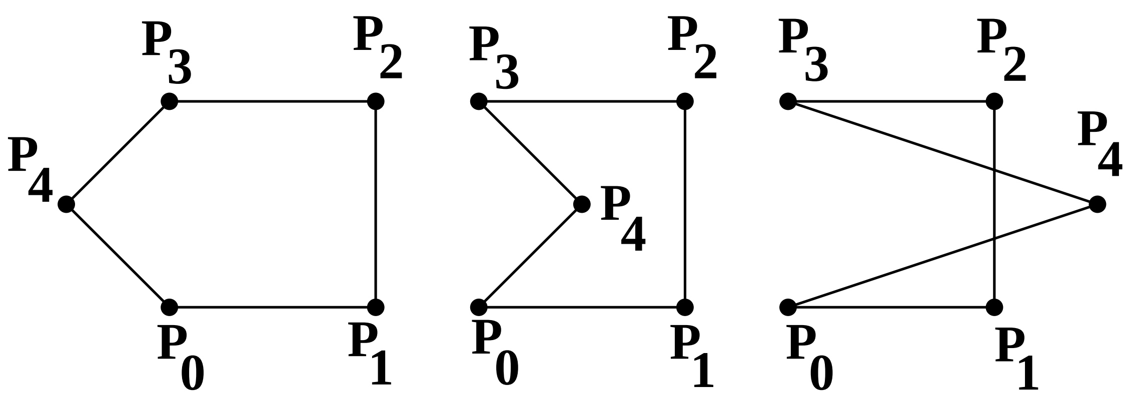


Figure 89: Pentagons–convex, non-convex and with self-intersection.

25.2 Regular polygons

There are many equivalent definitions for these most symmetric of all polygons.

(a) **Definition 17** *We shall say that a polygon $\mathcal{P}$ is* regular *if it has*

(i) all n vertices $P_0\ , \ldots, P_{n-1}$ on some circle λ, say with centre C and radius k; and

(ii) all edges of equal length x.

Thus a regular polygon is necessarily planar; we shall not pursue an interesting generalization which allows skew regular polygons. By the way, the circle λ is called the *circumcircle* for $\mathcal{P}$, and k is the *circumradius.*

(b) Some examples of regular polygons are the square $\{4\}$, the equilateral triangle $\{3\}$ and the regular pentagon $\{5\}$. Note, however, that a pentagram $\{5/2\}$ also satisfies our definition: its vertices clearly lie on a circle and its edges are of one and the same length. (Note that 'false crossings' such as B are *not* vertices of the polygon.)

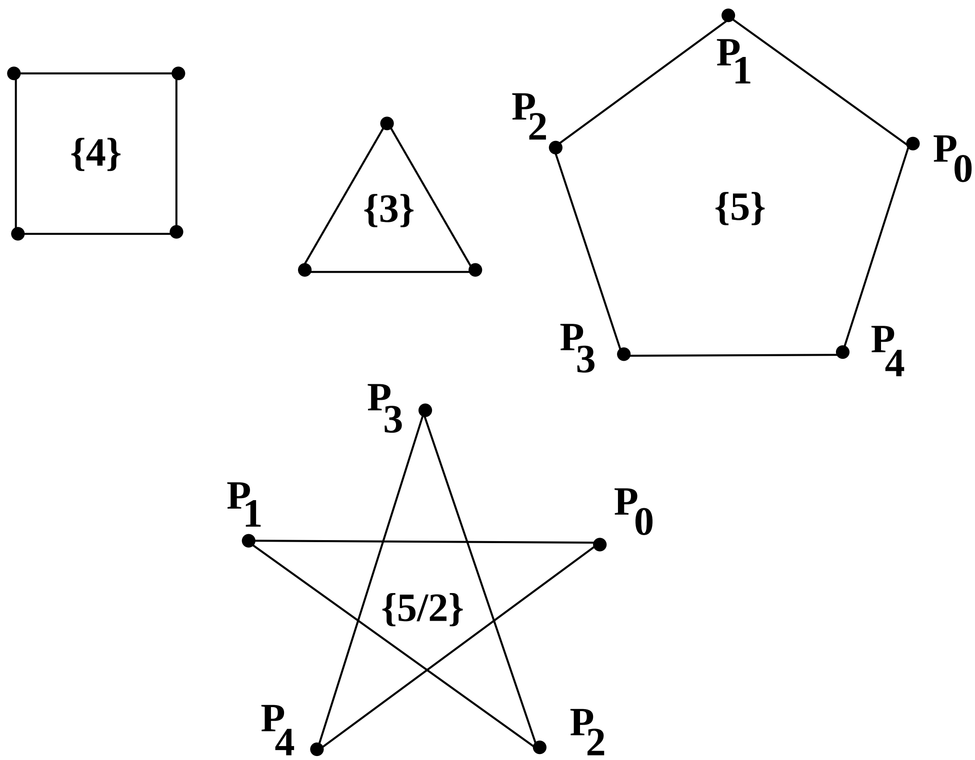


Figure 90: Typical regular polygons.

(c) *Metrical Properties of* $\mathcal{P}$ (Refer to Figure 91.) From the centre C drop perpendiculars to the consecutive edges $P_{j-1}P_j$ and P_jP_{j+1}. (Each edge is a chord of λ, so that midpoints F_{j-1} and F_j are are the feet of these perpendiculars.) Now $\sin\alpha = x/(2k)$, which is the same for every edge. Thus each edge subtends the same *central angle* at C; furthermore , we can calculate the *vertex angle* 2β since $\alpha + \beta = 90°$.

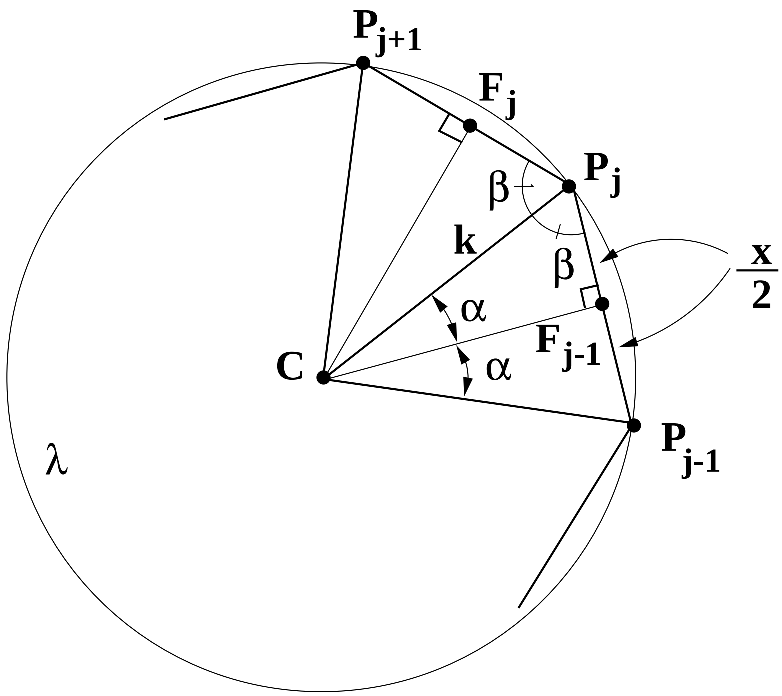


Figure 91: Metrical properties of regular polygons.

Conclusions:

(i) Each central angle $\angle P_{j-1}CP_j = 2\alpha$, where $\sin\alpha = x/(2k)$.

(ii) Each vertex angle $\angle P_{j-1}P_jP_{j+1} = 180° - 2\alpha$. (Thus, 2α is the external angle at each vertex.)

(d) We haven't yet related the measurements x and k to the number of vertices n in $\mathcal{P}$. Imagine that you start at P_0, move along the edge to P_1, then to P_2, and so on until you hit P_{n-1} then finally return to P_0. You have circled around the centre; but if you try this with the pentagram above you will see that you could wind around the centre C several times. The *winding number* w for $\mathcal{P}$ is the total number of times the centre is circled as one traverses the consecutive edges *once over*.

(i) Thus the total of the n equal central angles is

$$n(2\alpha) = w(360°)$$

so that the central angle is

$$2\alpha = \frac{360°}{n/w} \quad .$$

(ii) A regular polygon of this type is denoted $\{n/w\}$, as with the pentagram $\{5/2\}$. The polygon is convex precisely when the winding number $w = 1$ (or $n-1$ which you will find results in the same polygon). In this case we simply write $\{n\}$.

(iii) Notice that $1 \leq w < n$. It is also true that we may suppose that n and w have no common factors (other then 1, of course); we say that n and w are *relatively prime*. To convince yourself of this, sketch the regular polygon $\{6/2\}$. To do this you must draw six segments, while at the same time winding twice around the centre; by part (a) the central angle is $120° = 360°/(6/2)$. Does your polygon actually have six sides? How would you describe these six sides?

(e) As a further example, let's calculate the edge length of a dodecagram $\{12/5\}$ inscribed in the unit circle. Thus, $2(\alpha) = 360°/(12/5) = 150°$, so $\alpha = 75°$. Since $k = 1$, we have $\sin(75°) = x/2$, so the edge length is $x = 2\sin(75°) = \sqrt{2+\sqrt{3}}$. Likewise, the *inradius of* $\mathcal{P}$ (distance from C to the edge midpoints) is $\cos(75°) = \dfrac{\sqrt{2-\sqrt{3}}}{2}$.

25.3 Constructions of Regular Polygons

(a) If we already have drawn a regular n–gon $\{n\}$, then by connecting every second vertex we obtain $\{n/2\}$. Similarly, we can easily draw $\{n/w\}$, for any winding number w (where $1 \leq w < n$). But how do we draw the original convex n–gon $\{n\}$?

For small values of n we can use a protractor to lay out the required central angle; compasses can then be used to mark off the n equal edges. However, protractors are considered to be mathematically crude, so we now ask which regular polygons $\{n\}$ can be constructed using only ruler and compasses. This problem fascinated the ancient mathematicians; but it was not until 1837 that a complete answer was given.

(b) You should be familiar with elementary constructions for the square $\{4\}$ and equilateral triangle $\{3\}$. Now bisecting the central angles of $\{n\}$ gives (on the circumcircle λ) the n extra vertices needed to draw a $\{2n\}$. Hence, without much effort we can construct $\{6\}, \{12\}, \{24\}, \ldots$ and $\{8\}, \{16\}, \ldots$.

In fact, Euclid IV.11 and IV.16 describe constructions for $\{5\}$ and $\{15\}$, respectively. From these we can then construct $\{10\}, \{20\}, \ldots$ and $\{30\}, \{60\}, \ldots$. This was essentially the state of affairs from antiquity until 1796, when a 19-year-old German, *Carl Friedrich Gauss*, stunned the mathematical world by producing totally unexpected constructions for $\{17\}$ and other regular polygons.

(c) Gauss' discovery was particularly remarkable in that it linked the construction of $\{n\}$ to an important branch of number theory. In the seventeenth century, *Fermat* had investigated prime numbers of the form

$$F_k = 2^{2^k} + 1,$$

where $k \geq 0$. Thus, for example, $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$, all of which are prime numbers. However, the next Fermat number is $F_5 = 4294967297$, which is not prime. (It was first factored about a century ago: notice that we didn't actually claim above that F_k had to be prime!)

(d) Here then is *Gauss' construction*:
It is possible to construct the regular polygon $\{n\}$, using just ruler and compasses, when the only *odd prime factors* of n are *distinct* Fermat primes.

Thus, Gauss gave (in theory) constructions for $\{17\}$, $\{257\}$, $\{255\}$ (since $255 = 3{\cdot}5{\cdot}17$), etc. Of course, for larger values of n, an $\{n\}$ of moderate circumradius is virtually indistinguishable from a circle. Thus, the more interesting practical constructions are for $\{17\}$, $\{34\}$, $\{51\}$ and say $\{85\}$.

(e) We should carefully note that Gauss gave a *sufficient condition* for constructing $\{n\}$; conceivably, some totally different construction will allow us to construct $\{n\}$, when n has nothing to do with Fermat primes. However, as Gauss likely knew and P. L. Wantzel proved in 1837, there are no such n's. In other words, Gauss' condition on n is *necessary and sufficient* for the construction of the regular polygon $\{n\}$.

Hence, it is *impossible* to construct $\{11\}$, since 11 is not a Fermat prime. It is impossible to construct $\{9\}$, for in $9 = 3 \cdot 3$, the Fermat prime 3 is a repeated factor - note the word *distinct* in Gauss' condition.

Now to construct $\{9\}$, we would really only need to construct the central angle $40° = 360°/9$. Hence, we know it is *impossible* to construct with ruler and compasses a $40°$ angle.

In other words, although we can easily construct a $120°$ angle, we *cannot* trisect this angle. There are still many amateur mathematicians (and perhaps even a few misguided professionals) who vainly try to trisect general angles using only ruler and compasses. Even though some of these attempts are clever, they inevitably involve a misunderstanding or misuse of the rules of the construction game. [17]

25.4 Symmetry Groups

The regular polygon $\{n\}$ is clearly symmetric by rotation through $360°/n$ about the centre (and thus is symmetric by rotation through any multiple of this angle). Other symmetries include reflection in any line through the centre which also passes through either a vertex or an edge midpoint. (See Figure 90 for some typical cases.) Can you convince yourself that these isometries are indeed symmetries for $\{n\}$ and that there are no other symmetries?

How many symmetries are there? A cardboard model will help you understand the following argument. Fix one 'home' vertex for reference. This vertex can be moved to any of the n vertices (this includes staying put at home). Once there, we can either flip the polygon over or not. Hence there are altogether $2n$ distinct symmetries for $\{n\}$. This group of order $2n$ is called the *dihedral group D_n*. It therefore contains n rotations, including the identity, and n reflections.

Note that $\{n\}$ and $\{n/w\}$ have the same symmetry group (so long as n and w are relatively prime).

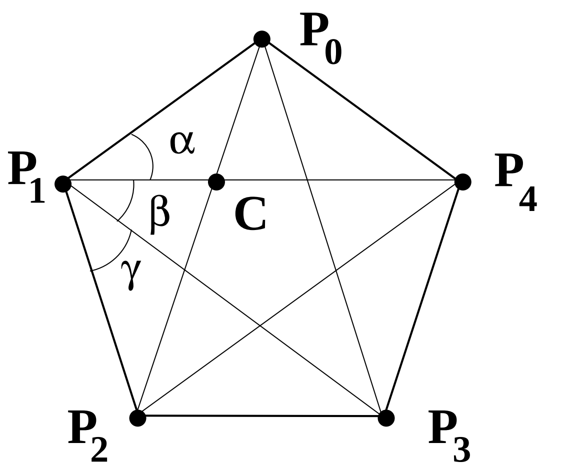
[17] There are ways to trisect angles if you change the construction rules, say by allowing a ruler with a couple of marks on it. But already 2000 years ago, Archimedes had such a trisection procedure.

25.5 Exercises on Regular Polygons

1. Construct an angle of $72°$. (Hint: It is known that $\cos 72° = (\sqrt{5}-1)/4$; see the next problem.)

2. *A proof that* $\cos(72°) = (\sqrt{5}-1)/4$.
   Here is a regular pentagon, with common edge length $P_{j-1}P_j = 1$. Notice that the five diagonals all have the same length and form an inscribed pentagram. Let's call this common diagonal length

$$\tau = P_0P_2 = P_1P_3,\ \ldots$$



   (a) Use a symmetry argument to show that $P_1P_4 \| P_2P_3$, and $P_0P_2 \| P_3P_4$. Hence show that $CP_2P_3P_4$ is a parallelogram.
   (b) Prove that $\triangle CP_0P_1 \sim \triangle CP_2P_4$.
   (c) Calculate τ after showing that

$$\frac{\tau - 1}{1} = \frac{1}{\tau} \ .$$

   (d) Show that $\alpha = \beta = \gamma = 36°$.
   (e) Prove that $\cos 36° = \tau/2$ and then calculate $\cos 72°$.

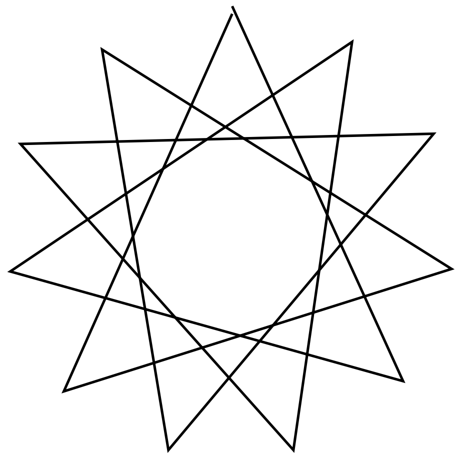
   You may check these results on your calculator!

3. Construct regular polygons with $4, 6, 8, 12, 5$ or 10 sides.

4. Using only a straightedge and compasses:
   (a) Construct a regular pentagon $P_1P_2P_3P_4P_5$.
   (b) Construct in the same circle an equilateral triangle with one vertex at P_1.
   (c) Thus construct a $\{15\}$.
   (Hint: By hand, make a rough sketch of a $\{15\}$. Observe the figure obtained by connecting every 3rd vertex, also by connecting every fifth.)

5. What is the order of the symmetry group for the regular dodecagon $\{12\}$? Briefly describe all its symmetries (a diagram may help).

6. What symbol $\{n\}$ should we associate with the polygon below?



7. For which $n \leq 20$, can the regular polygon $\{n\}$ be constructed using only compasses and straight-edge?

8. Draw $\{7\}$, $\{7/2\}$, $\{7/3\}$ in different colours and having a common vertex on the same circum-circle.

9. (a) For a regular polygon $\{n\}$ of edge length 2, compute the circumradius R for $n = 3, 4, 5, 6, 7, 8, 180$.

   (b) Take $n = 180$.

      i. Find the perimeter P of this $\{180\}$.

      ii. Since n is large, the polygon $\{180\}$ closely approximates a circle of radius R and circumference $2\pi R$. Thus $P \simeq 2\pi R$ so
      $$\pi \simeq \frac{P}{2R} \quad .$$

      iii. Compute π approximately by calculating $\dfrac{P}{2R}$.

   (c) (Viète's formula for π) Now and then we have casually mentioned the number π, without saying much concerning its definition. The reason is that a tightly rigorous description of π requires the idea of *limit* (from calculus) and thus a more serious exploration of foundations than pursued in these notes. Of course we can (and shall) define π in an elementary way using a circle. But this is a bit of a cheat since the notions of arclength and area themselves require calculus for a full development. Nevertheless, let us proceed in the intuitive way which is usually adopted in elementary geometry and see what we get.

   First of all, we may convince ourselves (in an elementary way), that for *any* two circles the ratios of circumference to diameter are equal. This allows us to make the following unambiguous definition:

Definition 18 *The number π is the ratio of the circumference to the diameter in a circle.*

Of course, we now have the usual formula $C = 2\pi r$, but this is little more than a rewording of the definition. However, one can now *prove* the significant fact that the *area* of any circle is given by $A = \pi r^2$, where π is the *same* constant as defined above.

The search for the actual 'value' of π has a long and complicated history. (See the chapter by D.E.Smith in reference [21].) For example, the implication of certain verses in the Bible is that $\pi = 3$, but this is only an approximation (about 4.5% smaller than the actual value 3.1415926...). Indeed, any *rational number* can be only an approximation, since it was proved in the late 18^{th} century that π is irrational. Thus, 22/7 is also a mere approximation, though a good one at about 0.04% too large.

In fact, we might say that π is unavoidably mysterious: since it is defined by some sort of limiting process, we probably should not expect a simple numerical or algebraic description. The purpose of the following exercises is to give a 'limiting formula' for π; the idea is due to the French mathematician François Viète (1540-1603).

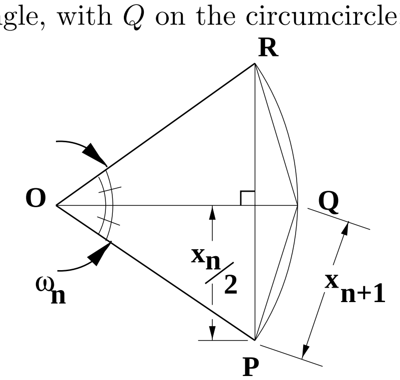
We shall look at the regular polygons $\{2^n\}$, inscribed in a unit circle with centre O, where $n = 1, 2, 3, \ldots$. Let the edge length be x_n; thus the perimeter of $\{2^n\}$ is $p_n = 2^n x_n$. Moreover, the central angle subtended by each edge is $\omega_n = 360°/(2^n)$.

i. Strictly speaking, the above setup makes sense only for $n > 1$, since a true polygon has more than $2 = 2^1$ sides. How should we interpret the polygon $\{2\}$, and what are the sensible values for its perimeter p_1 and edge length x_1 ?

ii. Let P and R be consecutive vertices of $\{2^n\}$, so that

$$\angle POR = \omega_n = \frac{360°}{2^n} \ .$$

Let OQ bisect this angle, with Q on the circumcircle:



Why are P and Q consecutive vertices of the polygon $\{2^{n+1}\}$? Show that $\angle QPR = (1/2)\angle POQ$; hence show that

$$\frac{p_n}{p_{n+1}} = \frac{x_n}{2x_{n+1}} = \cos \omega_{n+2} \ .$$

iii. For $n \geq 2$, show that

$$\frac{p_{n-1}}{p_n} = 2(\frac{p_n}{p_{n+1}})^2 - 1 \ .$$

Now express p_n/p_{n+1} in terms of p_{n-1}/p_n.

iv. Show that

$$\frac{p_1}{p_2} = \sqrt{\frac{1}{2}} \ .$$

Now verify Viète's formula for π:

$$\begin{aligned} \frac{2}{\pi} &= \lim_{n\to\infty} (\frac{p_1}{p_2})(\frac{p_2}{p_3}) \ldots (\frac{p_{n-1}}{p_n})(\frac{p_n}{p_{n+1}}) \\ &= \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \ldots \end{aligned}$$