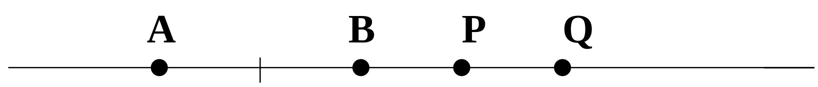
# Ratio, Area and Barycentric Coordinates

1. In many geometric problems, one encounters the **ratio**, like

$$\frac{AB}{PQ} \text{ or } [AB : PQ],$$

of two particular distances. When the corresponding **line segments**, which we also denote by $AB$ and $PQ$, lie on the same line $m$, it is convenient to agree that

the ratio $\dfrac{AB}{PQ}$ is positive, if the two segments have the same sense
on the common line $m$, or negative if they have opposite sense:

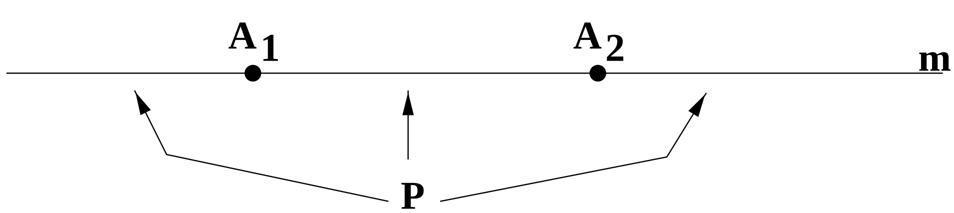



$$\frac{AB}{PQ} = +2, \quad \frac{PA}{BQ} = \frac{-3}{2}\ .$$

If $m$ is given a 'positive sense', or **orientation**, say by placing an arrow at one end, then each segment $AB$ on the line can be given a **signed distance**, according to whether the direction from $A$ to $B$ follows the sense of the line or goes against it. Note as well that if we change the unit of measurement on the line, the length of each segment is rescaled by some constant $\kappa$. However, no matter how we change either the sense or unit of measurement, the

ratio $\dfrac{AB}{PQ}$ is unchanged.

2. Suppose now that $A_1, A_2$ are any two distinct points (on a line, in the plane, in space, etc.). Let $P$ be any point on the line $m$ through $A_1$ and $A_2$:[1]




[1] If context is clear, we sometimes write $m = A_1A_2$ to indicate the whole line, rather than the segment or even the distance; this ambiguity makes for easier notation and is well worth the risk.

Notice that no matter where $P$ is located on the line (i.e. on the segment $A_1A_2$, or on either of the rays $A_1/A_2$ or $A_2/A_1$), we have

$$A_1P + PA_2 = A_1A_2$$

so that

$$\frac{A_1P}{A_1A_2} + \frac{PA_2}{A_1A_2} = 1.$$

(Keep in mind our convention that oppositely directed distances have opposite sign.)

**Definition** We say that $P$ has **barycentric coordinates**

$$x_1 = \frac{A_1P}{A_1A_2}\ ,\ \ x_2 = \frac{PA_2}{A_1A_2}$$

(with respect to the **affine basis** $A_1, A_2$), and we write

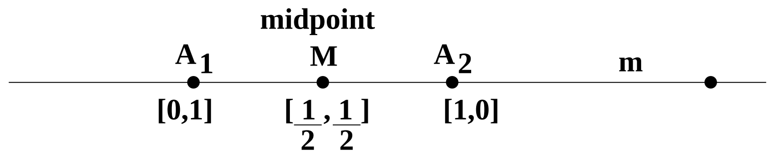
$$P = [x_1, x_2]$$

to indicate this.

Thus $P$, a point moving on a 1-dimensional line is described by 2 real numbers. These coordinates are therefore *redundant*. Indeed, they satisfy

$$x_1 + x_2 = 1.$$

Here are some typical values:




| | $A_1$ | M | $A_2$ | m |
|---|---|---|---|---|
| $x_1$ | 0 | 1/2 | 1 | 2 |
| $x_2$ | 1 | 1/2 | 0 | -1 |
| $r = \frac{x_1}{x_2}$ | 0 | 1 | $\infty$ | -2 |

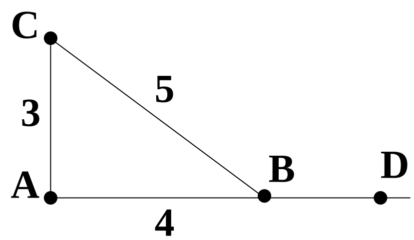
The ratio $r = \dfrac{x_1}{x_2} = \dfrac{1-x_2}{x_2}$ can also be used to coordinatize points on the line, though we must naturally use the symbol $r = \infty$ to locate the point $A_2 = [1, 0]$. This sort of coordinate is most useful when we generalize ordinary geometry to **projective geometry**.

3. The above ideas generalize in a very useful way to the plane (and even to higher dimensions).

   **Definition** If $A, B, C$ are any three points in the plane we define the **signed area** (or just area) $(ABC)$ as follows:

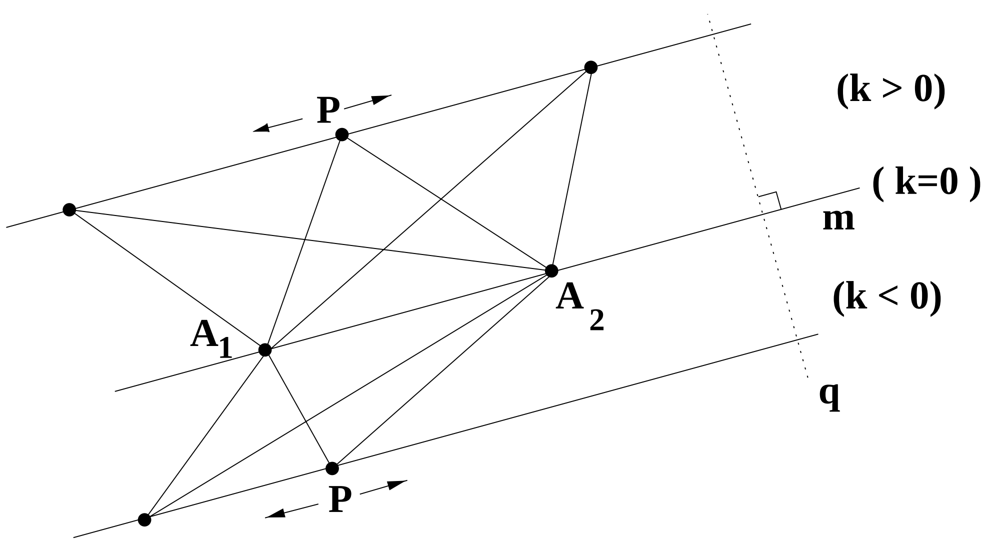
$$(ABC) := \begin{cases} \text{area of } \triangle ABC, & \text{if oriented anti-clockwise} \\ -\ \text{area of } \triangle ABC, & \text{if oriented clockwise} \\ 0, & \text{otherwise (if } A, B, C \text{ lie on one line).} \end{cases}$$

   Here is a typical example:




$$\begin{array}{ccccccc} (ABC) & = & (BCA) & = & (CAB) & = & 6 \\ (BAC) & = & (CBA) & = & (ACB) & = & -6 \\ (ABD) & = & (ABB) & = & (BAD) & = & 0\,. \end{array}$$

4. Now fix two distinct points $A_1, A_2$ in the plane and let $P$ be a (variable) point free to move along any line parallel to $m = A_1A_2$:




   In all cases, the signed area $(PA_1A_2)$ is constant as $P$ moves along a line *parallel* to $A_1A_2$, since such triangles have a common base and constant height (as measured along a common perpendicular $q$ to the parallel lines). We conclude that the equation

$$(PA_1A_2) = k \tag{1}$$

   defines a line parallel to $m = A_1A_2$. In particular, the line $A_1A_2$ itself has equation

$$(PA_1A_2) = 0, \tag{2}$$

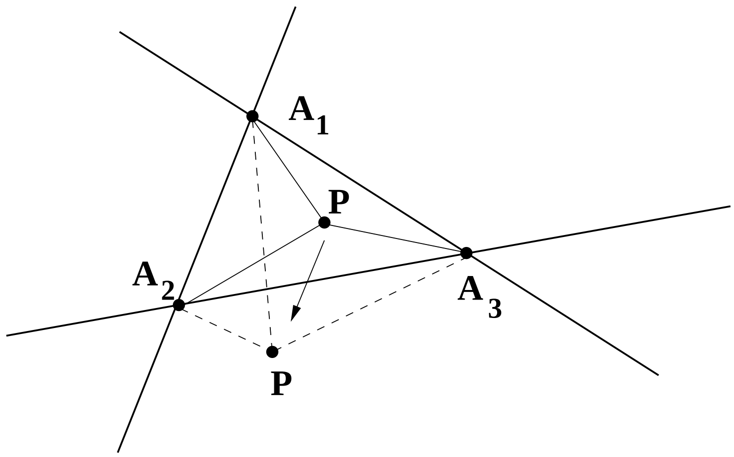
and splits the plane into two **open half-planes**, described by the inequalities $(PA_1A_2) > 0$ and $(PA_1A_2) < 0$, respectively.

Notice that if we move $P$ so that it approaches, then crosses, line $A_1A_2$ to the other side, the area $(PA_1A_2)$ switches sign by passing through 0.

5. **Theorem 1**

Let $\triangle A_1A_2A_3$ be any triangle in the plane. Then for any point $P$ in the plane we have

$$(PA_2A_3) + (PA_3A_1) + (PA_1A_2) = (A_1A_2A_3).$$



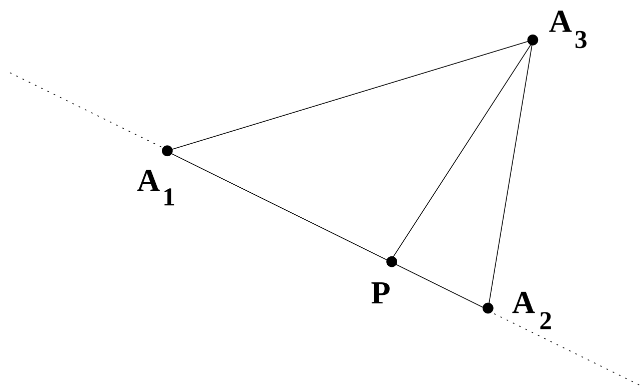

**Proof**. When $P$ is inside $\triangle A_1A_2A_3$, the three (signed) areas clearly total to the area of $\triangle A_1A_2A_3$. Now observe that when $P$ crosses any (extended) side, say $A_2A_3$, the area $(PA_2A_3)$ changes orientation, and hence sign, so that the area sum of the theorem remains fixed at $(A_1A_2A_3)$. □

Remark. The result is true if $\triangle A_1A_2A_3$ has clockwise orientation, or even degenerates.

**Theorem 2**

If $P$ is any point on side $A_1A_2$ of $\triangle A_1A_2A_3$, then

$$\frac{(A_1PA_3)}{(PA_2A_3)} = \frac{A_1P}{PA_2}.$$




6. We already have enough machinery to prove a remarkably useful theorem due to the 17th century Italian mathematician, Giovanni Ceva.

   **Definition.** A **cevian** of a triangle $\triangle ABC$ is a line segment joining a vertex to a point on the opposite side (perhaps extended).

   We ask when three cevians $AX, BY, CZ$ are **concurrent** (i.e. pass through one point $P$). Of course, this is not usually the case.

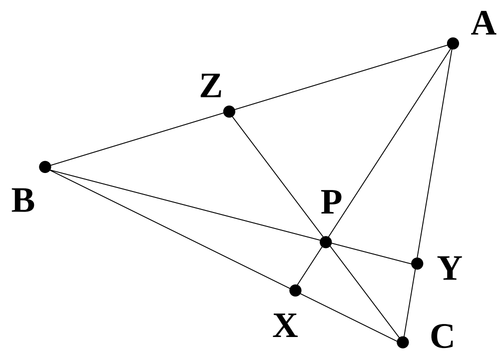
   **Theorem 3**

   Three cevians $AX, BY, CZ$ in $\triangle ABC$ are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

   Remark: These are ratios of directed segments, so for example $BX = -XB$. Thus it is important to write the segments exactly as indicated (at least in pairs!). To this end, note how the six segments are encountered during a circuit of the edges of the triangle.

**Proof.** Suppose $AX, BY, CZ$ meet at $P$, as below:




Then by previous theorems we get

$$\frac{BX}{XC} = \frac{(BXA)}{(XCA)} = \frac{(BXP)}{(XCP)} = \frac{(BXA) - (BXP)}{(XCA) - (XCP)}$$

so

$$\frac{BX}{XC} = \frac{(BPA)}{(APC)} \ ,$$

and similarly

$$\frac{CY}{YA} = \frac{(CPB)}{(BPA)} \ , \quad \frac{AZ}{ZB} = \frac{(APC)}{(CPB)} \ .$$

Multiplying these equations we get

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

The converse follows quite easily from the first half of the theorem. □

7. Let us again fix a (proper) triangle $\triangle A_1A_2A_3$ in the plane, so that $(A_1A_2A_3) \neq 0$. For each point $P$ in the plane we have proved (Theorem 1) that

$$(PA_2A_3) + (PA_3A_1) + (PA_1A_2) = (A_1A_2A_3) \; ,$$

so that

$$\frac{(PA_2A_3)}{(A_1A_2A_3)} + \frac{(PA_3A_1)}{(A_2A_3A)} + \frac{(PA_1A_2)}{(A_3A_1A_2)} = 1.$$

**Definition.** The point $P$ has **barycentric coordinates**

$$x_1 = \frac{(PA_2A_3)}{(A_1A_2A_3)} \; , \quad x_2 = \frac{(PA_3A_1)}{(A_2A_3A_1)} \; , \quad x_3 = \frac{(PA_1A_2)}{(A_3A_1A_2)} \; ,$$

and we write $P = [x_1, x_2, x_3]$.

Again we note that such coordinates depend on the triangle of reference $\triangle A_1A_2A_3$, and are redundant, since

$$x_1 + x_2 + x_3 = 1.$$

It is quite clear that the vertices of the reference triangle have coordinates

$$A_1 = [1, 0, 0], \quad A_2 = [0, 1, 0], \quad A_3 = [0, 0, 1].$$

Furthermore, by observation (1) on page 3, any line parallel to side $A_2A_3$ has an equation of the form

$$\begin{aligned} x_1 &= k \\ &= k(x_1 + x_2 + x_3), \end{aligned}$$

or

$$(k-1)x_1 + kx_2 + kx_3 = 0,$$

thus, a *linear homogeneous equation.*

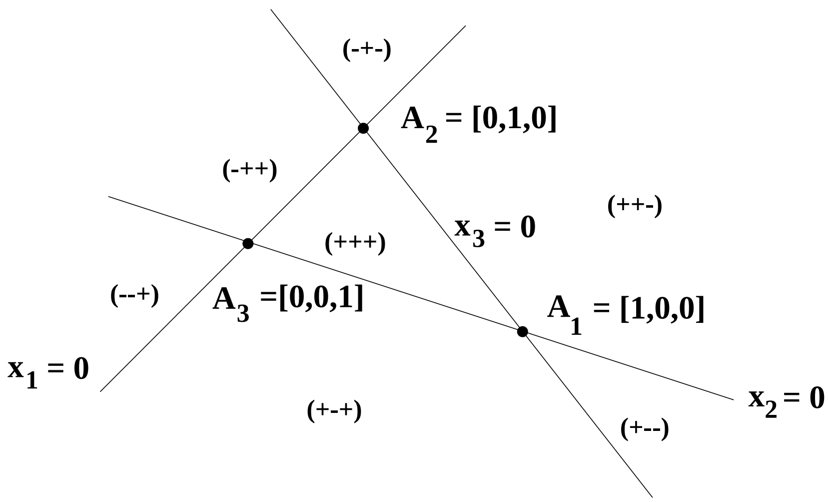
In particular, the (extended) side $A_2A_3$, in which $k = 0$ (see (2) on page 3), has the equation

$$x_1 = 0.$$

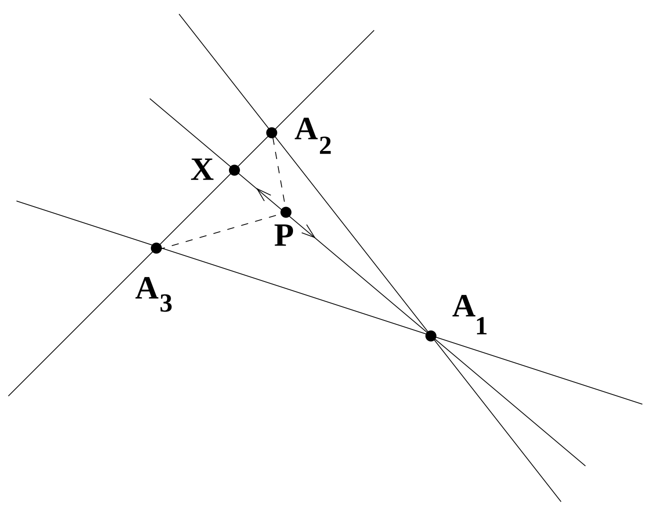
In brief then, the line opposite $A_j$ has equation $x_j = 0$. These lines split the plane into 7 non-overlapping open regions, each of which can be specified by the sign pattern

$$(+++), \ldots, (--+)$$

of its coordinates. Only one sign pattern is prohibited by the constraint $x_1 + x_2 + x_3 = 1$. (Which?)




Cevians for $\triangle A_1A_2A_3$ also have rather simple equations. Consider, for example, the line $A_1X$, where $X = [0, a, 1 - a]$ is some fixed point on line $A_2A_3$:



Then by Theorem 2,

$$
\begin{aligned}
\frac{a}{1-a} &= \frac{(A_2XA_1)}{(XA_3A_1)} \\
&= \frac{(A_2XP)}{(XA_3P)} \\
&= \frac{(A_2XA_1)-(A_2XP)}{(XA_3A_1)-(XA_3P)} \\
&= \frac{(PA_1A_2)}{(PA_3A_1)} \\
&= \frac{x_3}{x_2}.
\end{aligned}
$$

Thus the cevian $A_1X$ has the homogeneous linear equation

$$0 = 0x_1 + ax_2 + (a-1)x_3.$$

In particular, the midpoint of $A_2A_3$ is $\left[0, \frac{1}{2}, \frac{1}{2}\right]$, so that the **median** from $A_1$ has equation $x_2 = x_3$. Likewise, the other medians are $x_1 = x_3$ and $x_1 = x_2$; and the three medians meet in the **centroid**

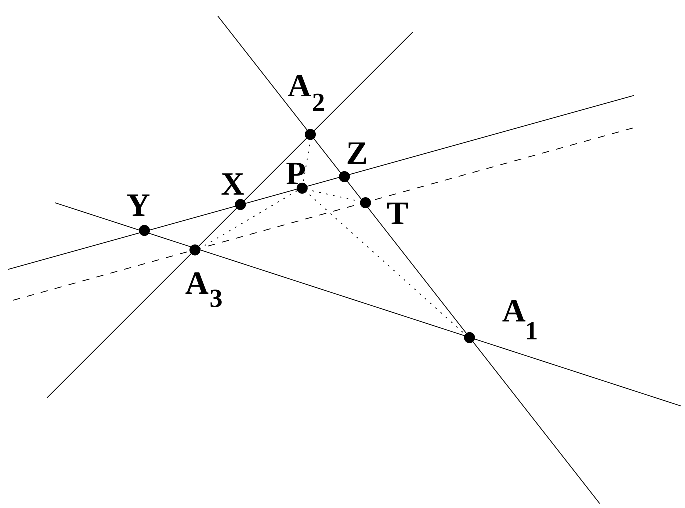
$$G = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$$

of $\triangle A_1A_2A_3$. We note that the three medians split the triangle into 6 smaller triangles of equal area.

In fact, any line $\ell$ in the plane is described by a linear homogeneous equation. We have already verified this when $\ell$ passes through a vertex or is parallel to a side of $\triangle A_1A_2A_3$. Thus we may assume that $\ell$ meets the (extended) sides in three distinct points, say

$$
\begin{array}{lcll}
X &=& [0, a, 1-a] & \text{on} \quad A_2A_3 \\
Y &=& [1-b, 0, b] & \text{on} \quad A_1A_3 \\
Z &=& [c, 1-c, 0] & \text{on} \quad A_1A_2.
\end{array}
$$

(Our assumption on $\ell$ amounts to assuming that $abc(1-a)(1-b)(1-c) \neq 0$.)



Now find $T$ on $A_1A_2$ so that $A_3T$ is parallel to $\ell$. Note that as $P$ varies on $\ell$, both $(PA_3T)$ and $(A_3A_1T)$ are constant, say with sum

$$
\begin{aligned}
k &= (PA_3A_1T) \\
&= (PA_3T) + (A_3A_1T) \\
&= (PA_3A_1) + (A_1TP) \\
&= \frac{(PA_3A_1)}{(A_2A_3A_1)}\,(A_2A_3A_1) + \frac{(A_1TP)}{(A_1A_2P)}\,\frac{(A_1A_2P)}{(A_1A_2A_3)}\,(A_1A_2A_3) \\
&= x_2 \cdot m_1 + x_3 \cdot m_2,
\end{aligned}
$$

for certain constants

$$m_1 = (A_1A_2A_3)\,, \quad m_2 = (A_1A_2A_3)\,\frac{A_1T}{A_1A_2}\,.$$

Since $x_1 + x_2 + x_3 = 1$ we get the equation

$$kx_1 + (k - m_1)x_2 + (k - m_2)x_3 = 0$$

for $\ell$.

Collecting all these results, we obtain the following

**Theorem 4**

Each line in the plane is described by a linear homogeneous equation of the form

$$0 = k_1x_1 + k_2x_2 + k_3x_3 \qquad (x_1 + x_2 + x_3 = 1)$$

for constants $k_1, k_2, k_3$ not all 0.

8. The base triangle $\triangle A_1A_2A_3$ is quite arbitrary. Referring again to the intercepts $X, Y, Z$ on the line $\ell$, we find that

$$\frac{A_2X}{XA_3} \cdot \frac{A_3Y}{YA_1} \cdot \frac{A_1Z}{ZA_2} = \frac{1-a}{a} \cdot \frac{1-b}{b} \cdot \frac{1-c}{c}\,.$$

Now the points $X, Y, Z$ are collinear if and only if they lie on some line $k_1x_1 + k_2x_2 + k_3x_3 = 0$ (not all $k_j = 0$), that is to say, if and only if the matrix equation

$$\begin{bmatrix} 0 & a & 1-a \\ 1-b & 0 & b \\ c & 1-c & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a non-trivial solution in the $k_j$'s. But this happens precisely when

$$0 = \begin{vmatrix} 0 & a & (1-a) \\ 1-b & 0 & b \\ c & 1-c & 0 \end{vmatrix} = abc + (1-a)(1-b)(1-c),$$

that is when

$$\frac{1-a}{a} \cdot \frac{1-b}{b} \cdot \frac{1-c}{c} = -1.$$

We have therefore verified

**Theorem 5**

(**Menelaus**) Points $X, Y, Z$ on the (extended) sides

$$A_2A_3,\ A_1A_3,\ A_1A_2 \ \text{ of } \ \triangle\, A_1A_2A_3$$

are collinear if and only if

$$\frac{A_2X}{XA_3} \cdot \frac{A_3Y}{YA_1} \cdot \frac{A_1Z}{ZA_2} = -1.$$