Some Beautiful Properties of Circles

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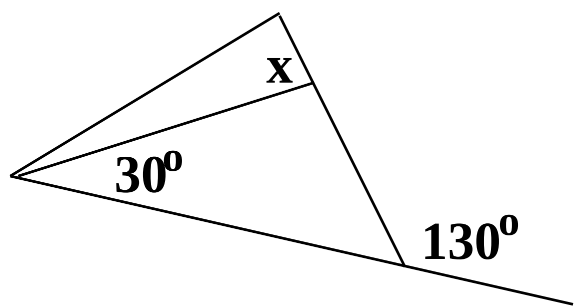
1 Why and how do we prove things in mathematics?

Well, that is a big question, and this sentence is the short answer. (See § 9 at the end for a more leisurely discussion.)

1. Anyway, let's warm up with

   Exercise 1: Find the angle labelled x in

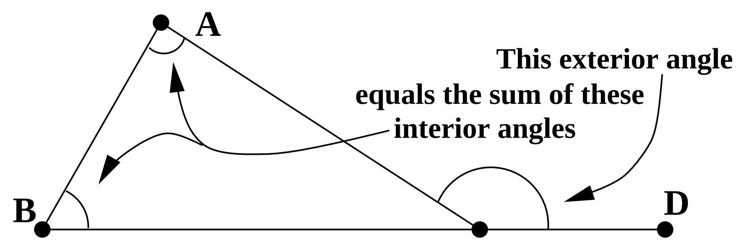
   

   Your solution:

2. We can speed up our calculation with a little theorem I want you to prove. (Maybe you have seen it before.) Since this theorem concerns ‘exterior angles’, we give it the convenient abbreviation [**ext.** $\angle$].

   Theorem. [**ext.** $\angle$] In any triangle, an exterior angle is the sum of the two opposite interior angles:

$$\begin{aligned} \angle ACD &= \angle BAC + \angle CBA \\ &= \angle A + \angle B \qquad \text{(here this is unambiguous!)} \end{aligned}$$



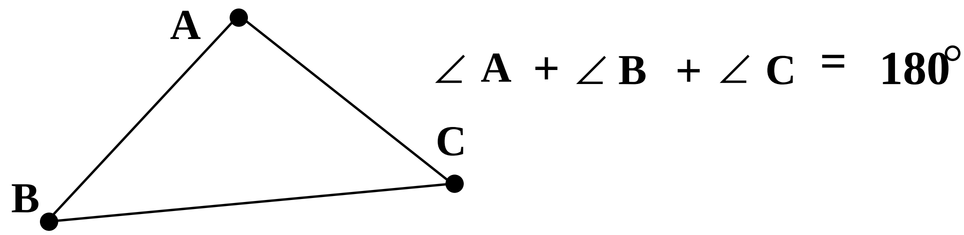
   Your proof:

   My proof (may be useful to compare)

3. **Exercise 2:** Use [**ext.** $\angle$]to do Exercise 1 in your head.

4. Before constructing some new geometry, we'll dig a little into its foundations. Let's look critically at my proof.

   (a) Why do the angles in a triangle sum to 180°?

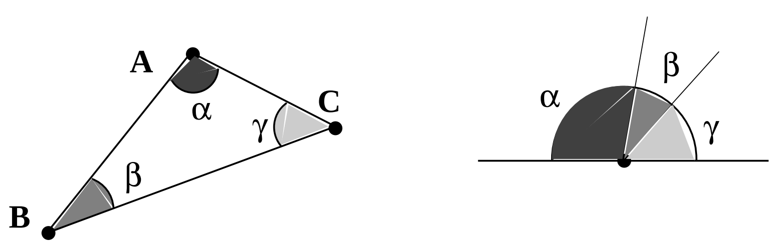


   How could we prove this? And what is a 180° angle anyway? Actually, although degrees are a convenient way to measure angles, there is nothing special about 180 – we could have chosen 170, so getting slightly bigger units.

   (b) **Exercise 3:** What is so good about 180 ?

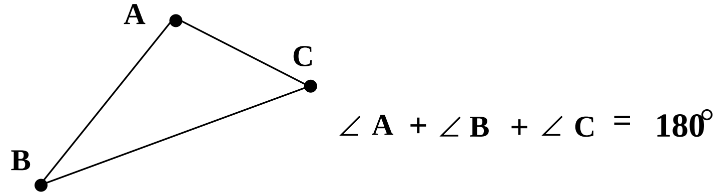
   (c) In fact, we can avoid this issue: by a 180° angle we really mean a straight angle $\angle BCD$ formed by three consecutive points on a line:

   (d) So our angle sum theorem really asserts something more interesting – if you cut off the angles of a triangle, they will fit together perfectly to fill a straight line:

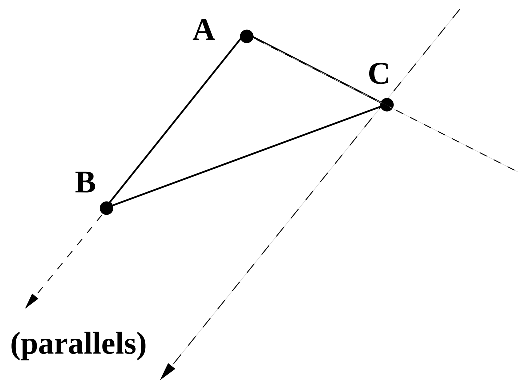


(e) So how can we use this idea to prove our angle sum theorem?

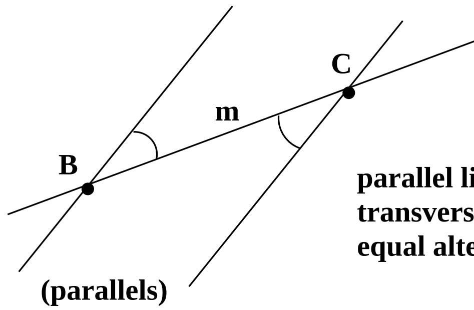
Theorem [$\angle$ **sum in** $\triangle$]: The angles in $\triangle ABC$ sum to a straight angle:



My proof:



(f) Why do parallel lines meet the line m in equal alternate angles? **And what are parallel lines anyway?**



parallel lines cut the transverse line m in equal alternate angles

You see that we could continue this game! But eventually we must stop our definitions and accept certain primitive terms as being undefined (but usually we do use words like **point**, **line** with which we are comfortable).

(g) Ultimately, we confront the Euclidean behaviour of parallel lines at the bottom of our mathematical foundations. It is here that we can, if we wish, enforce a different kind of behaviour and so create **non-Euclidean geometry**.

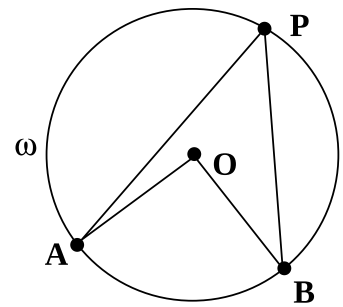
Instead, let's move forward into the Euclidean world of circles!

2 Angles inscribed in circles.

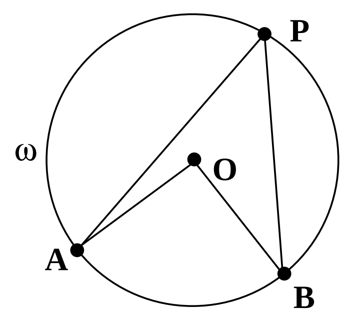
1. Now, with your help, let's prove:

   Theorem [$\angle$ **at arc**] Suppose A, B are two points on a circle ω with centre O, and let P be any point on the arc (not subtended by $\angle AOB$). Then

   $$\angle APB = \frac{1}{2}\angle AOB.$$

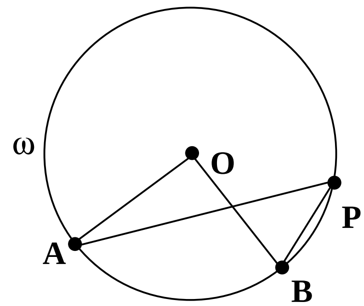
   

   Our proof: Where to start? What do we have? Where must we go?

   (See § 7 below for abbreviations of commonly used results.)

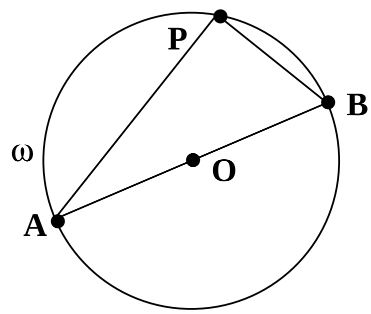
2. Actually, our proof above is valid only when P lies between AO (extended) and BO (extended). So there is a second case, which can and must be proved.



Your task: prove in this second case that $\angle APB = \frac{1}{2}\angle AOB$.

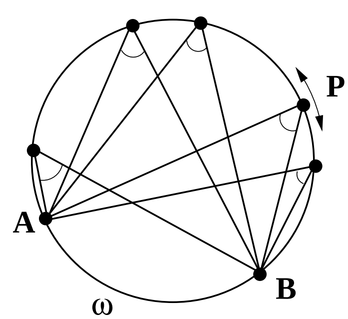
3. Complete the statements of these 'mini-theorems', or corollaries, and supply brief proofs.

   (a) **Corollary 1** [$\angle$ **in semi** ◯]: The angle in a semicircle is _____________$^\circ$; i.e., if AOB is a diameter, then $\angle APB =$_____________$^\circ$.

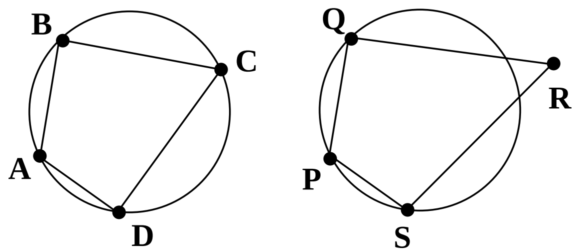


   (b) **Corollary 2.** Suppose A, B are two fixed points on a circle ω. Then as P varies on either of the arcs determined by A, B

$\angle APB$ is __________________________.

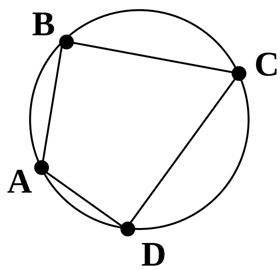


A **cyclic quadrilateral** is a 4-sided polygon which can be inscribed in a circle. Not every quadrilateral is cyclic.



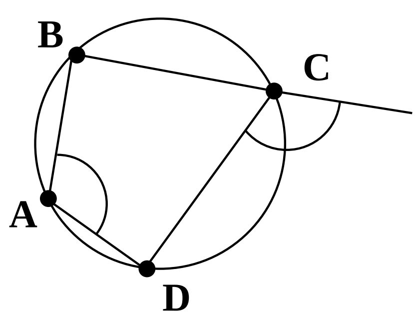
ABCD is cyclic **PQRS is non-cyclic**

(c) **Corollary 3** [**∠'s in cyc. quad**]: The opposite angles of a cyclic quadrilateral are supplementary (i.e., add to 180°): $\angle A + \angle C = 180^\circ = \angle B + \angle D$.



$$\angle \mathbf{A} + \angle \mathbf{C} = \mathbf{180^\circ} = \angle \mathbf{B} + \angle \mathbf{D}$$

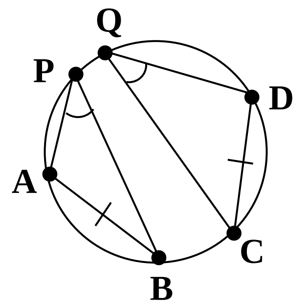
(d) **Corollary 4.** Any angle external to a cyclic quadrilateral equals the opposite internal angle.



(e) **Corollary 5.** Suppose $\triangle ABP$ and $\triangle CDQ$ are inscribed in a circle ω, and $AB = CD$. Then either

(i) $\angle APB = \angle CQD$ (when?); or

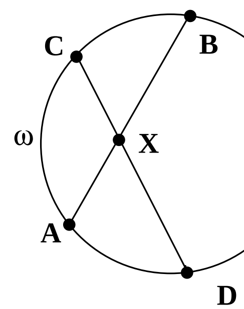
(ii) $\angle APB = 180° - \angle CQD$ (when?).



4. **Problems 4**: Try any of exercises 1–12 in § 8 below.

3 Product of segments on chords.

1. **Theorem [chords through int. pt.]**: Suppose AXB and CXD are two chords of a circle ω, which intersect at an interior point X. Then $XA \cdot XB = XC \cdot XD$.



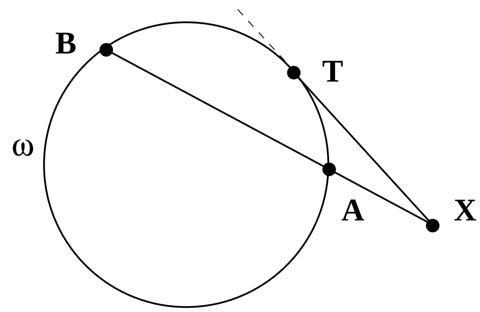
   Proof:

2. **** Some Important Jargon **** If X is interior to ω, the product $XA \cdot XB$ is independent of the chord AXB you choose to compute with. That is, with respect to the circle ω

   $XA \cdot XB$ is an **invariant** of the point X.

3. Let's move outside.

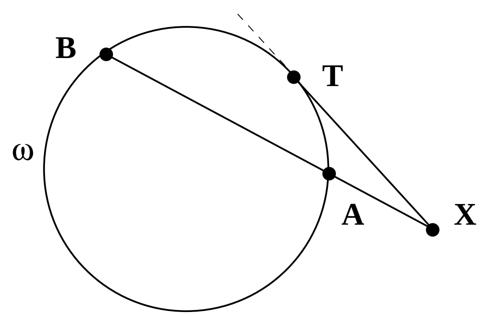
   Theorem [sec. and tang. from ext. pt.]: Suppose X is external to circle ω and XT is one of the two tangents to ω from point X:



   Suppose XAB meets ω at A, B. Then

$$XA \cdot XB = (XT)^2.$$

   Proof:



4. **Corollary.** Suppose X is external to circle ω and XAB, XCD are any two secants meeting ω at A, B and CD respectively. Then

$$XA \cdot XB = XC \cdot XD.$$

   (Thus this product is an **invariant** for the external point X.)

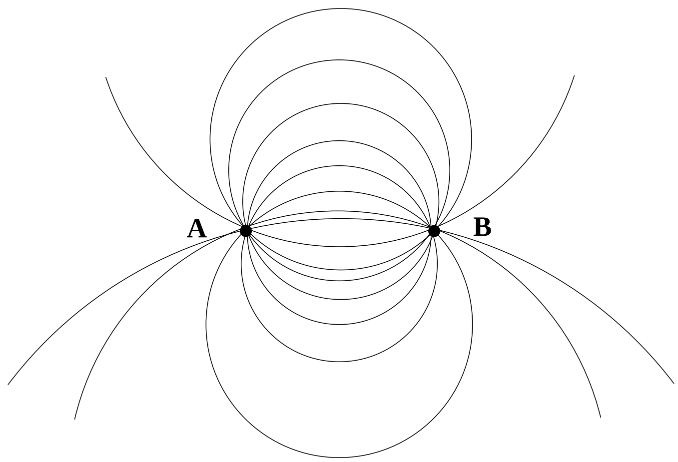
4 Pencils of Circles.

Although the ideas coming up are Euclidean in nature, they also are related, on a deeper level, to certain motions and families of curves in

non-Euclidean geometry.

(The families of circles shown below are useful in describing a Euclidean road-map, or model, of a non-Euclidean world.)

1. Look at the collection $\mathcal{I}$ of all circles through two fixed (and distinct) points A, B:



   The collection $\mathcal{I}$ is called an **intersecting pencil of circles**.

2. **Problems 5**:

   (a) Characterize the centres of the circles in $\mathcal{I}$.

(b) **A Coordinate Description.** Suppose

$$A = (-1, 0), \quad B = (1, 0).$$

(i) What are the centre and radius of a typical circle ω in the pencil $\mathcal{I}$?

Centre =

Radius =

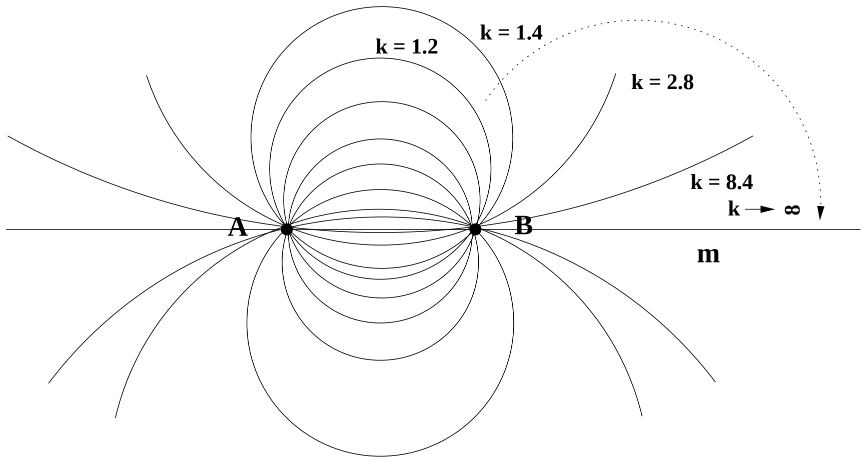
(A general answer has to involve a **parameter** k, which is fixed for each circle, but which varies somehow for different circles.)

(ii) What is the equation of a typical circle?

(c) Since k is free to vary, we may say that $\mathcal{I}$ is a **one-parameter family of circles**. As k becomes vary large, indicated by writing

$$k \to \infty$$

what happens to the circle ω?

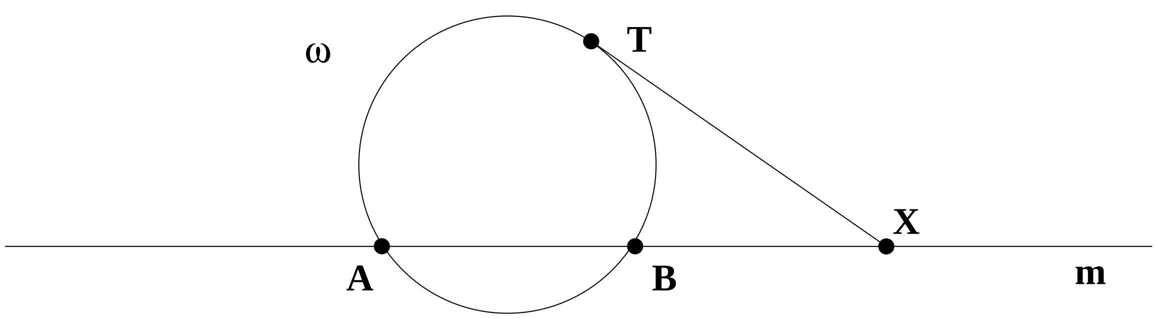


3. Thinking about this, we agree that it is very natural to include the (infinite) line m through A, B in the pencil $\mathcal{I}$. Some observations:

   - The straight line m fills a gap: with it, we see that with two exceptions, every point in the plane lies on exactly one member of the family $\mathcal{I}$.

     What are the two exceptional points?
   - The line m (or any line, by the same reasoning) is a circle of infinite radius, and whose centre is some point at infinity.
   - From the point of view of **inversive geometry**, every line is a 'circle of infinite radius, whose centre is the point ∞ (at infinity).

4. Now fix any point X on m, but outside segment AB. Draw the line XT tangent to some circle ω of the pencil.

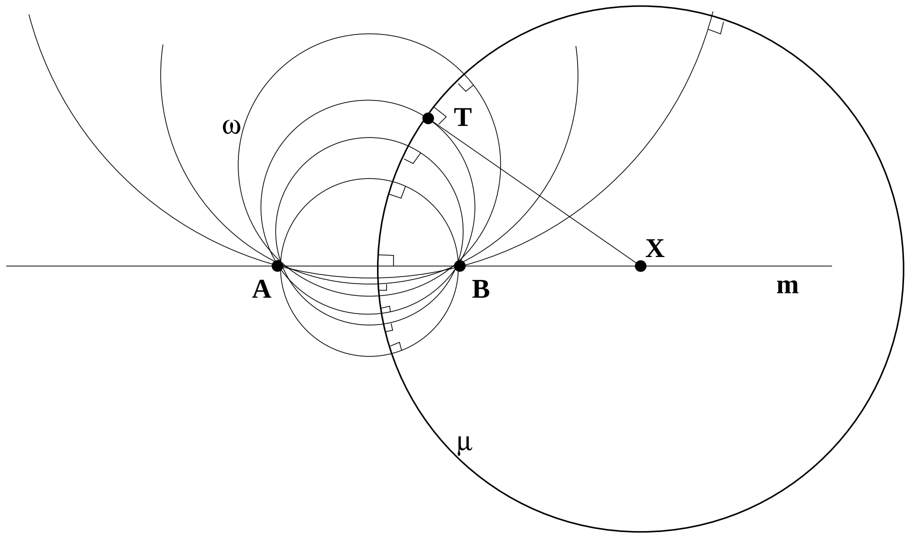


Thus

$$XT = \sqrt{XA \cdot XB}$$

which is independent of the circle $\omega \in \mathcal{I}$ (since all circles $\omega \in \mathcal{I}$ do pass through A, B).

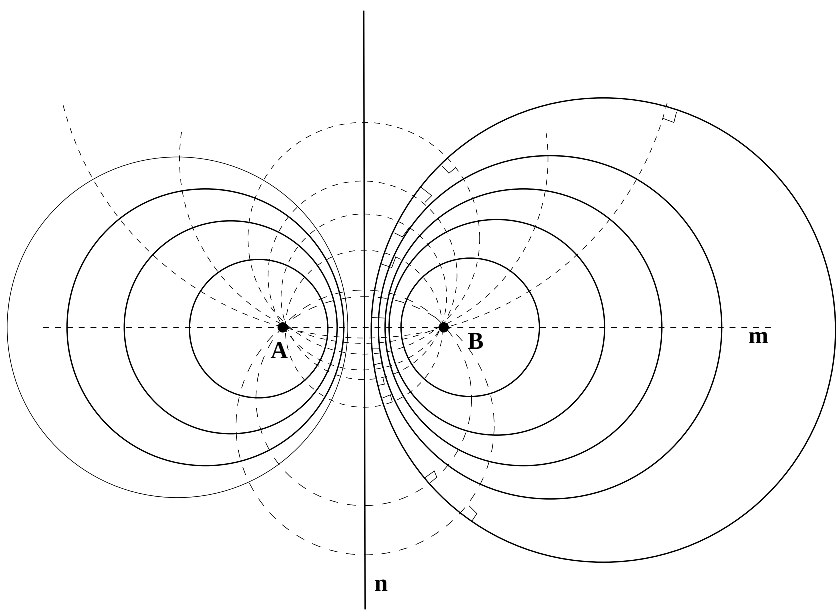
So as ω varies in the pencil $\mathcal{I}$, XT is nevertheless constant, hence T will trace out a circle μ, centred at X, a circle which is **perpendicular to every circle** ω of $\mathcal{I}$ (why?!):



To recap, for every point X as centre, we get a very special circle μ. These new circles μ taken together form

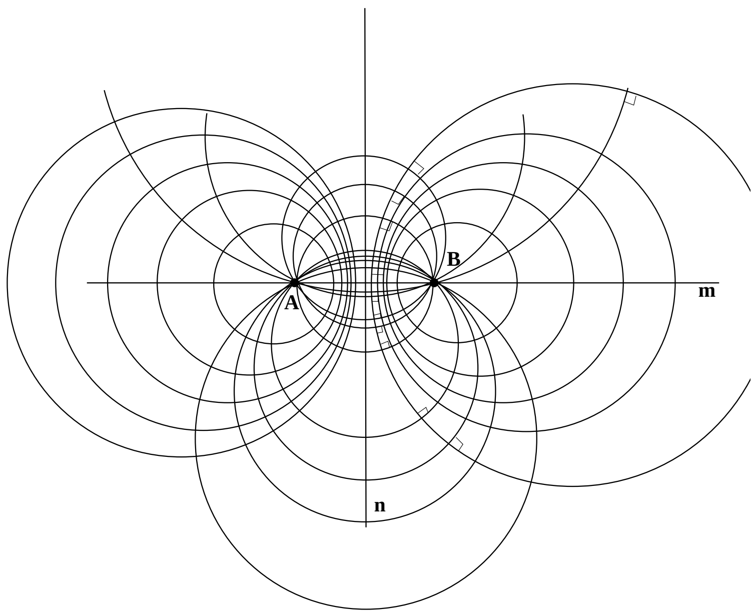
'the non-intersecting pencil $\mathcal{N}$'.

Again we include in $\mathcal{N}$, as one limiting case, the line n which is the perpendicular bisector of segment AB. At the other end of the scale, one can think of the limit points A, B as circles of radius 0 in $\mathcal{N}$:



5. **Exercise 6.** Explain why no two circles of $\mathcal{N}$ intersect.

6. Here is a picture of both pencils:



   Again observe that each circle ω of $\mathcal{I}$ meets each circle μ of $\mathcal{N}$ and does so in a perpendicular, or **orthogonal**, manner. That is, the tangents at any intersection point Q are perpendicular lines. By the way, this is how we measure angles between curves, like these circles, which intersect.

   We may say finally that $\mathcal{I}$ and $\mathcal{N}$ are **orthogonal pencils of circles**.

7. **Exercise 7.** What happens when we let A, B approach one another? If $A = B$, then we get two **mutually orthogonal tangent pencils**:

5 The Power of a Point.

Suppose ω is a fixed circle of radius R and centre O. If X is any point distant d from the centre, we call

$$d^2 - R^2$$

the **power** of X (with respect to the circle ω).

1. **Exercise 8.** When is the power of X positive? negative? zero?

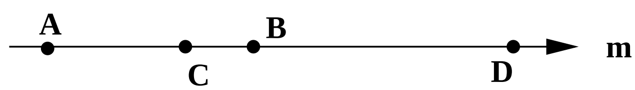
2. **Exercise 9.** What is the smallest value that the power of X can have with respect to a <u>fixed</u> circle whose radius is R? Which point has this extreme power?

3. Before continuing with this, we need a useful idea.

   The length of a segment AB is normally taken to be a positive number (or 0 if the end points coincide).

   However for points A and B on a **directed line** m, it is very useful to agree that

   - AB is positive, if A, then B, occur in the chosen direction along m,
   - AB is negative, if A then B occur against the chosen direction,
   - AB is zero, if $A = B$.



Thus	AB, CD, CB	are all	$\oplus$
	BC, DA	are all	$\ominus$
	$AB \cdot BC$	is	$\ominus$
	$AC \cdot BD$	is	$\oplus$.

   Notice that we could have originally directed m in the opposite sense (i.e., left). Then every individual length is reversed. For example, AB would be $\ominus$, BC $\oplus$, but still

$$AB \cdot BC \text{ would be } \ominus .$$

   (Recall $(-1)(-1) = +1$.)

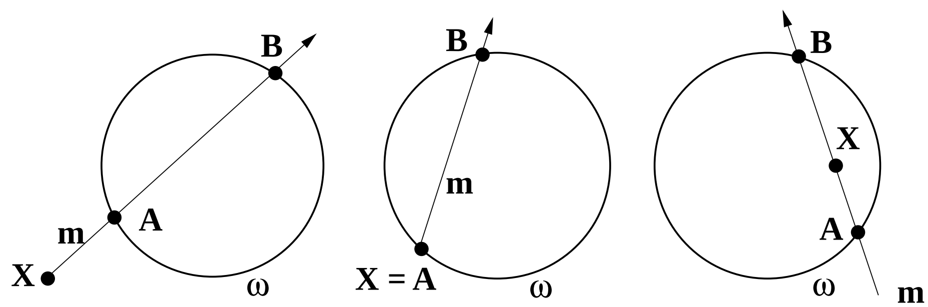
   <u>Conclusion:</u> The products of two directed lengths on a directed lines m is unchanged if we reverse the direction of m.

   This is useful, since we need not worry how we choose this direction.

4. Now we are ready for a result which ties all this together.

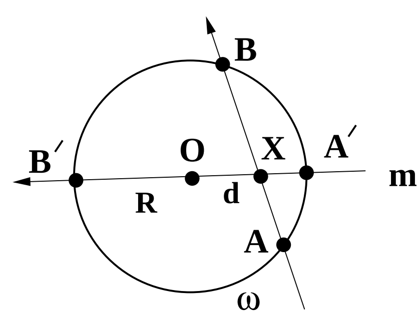
   Theorem. Fix a circle ω. For any point X in the plane suppose that a directed line m through X meets the circle in points A, B. (We may take $A = B$ if m is tangent to ω.) Then the power of X with respect to ω is

$$XA \cdot XB.$$



   Proof: The upshot of our big theorems in § 4 is that $XA \cdot XB$ is independent of our choice for line m. Note that when X is interior to ω, the product $XA \cdot XB$ is now considered to be negative.

   Anyway we can choose m to be the line through X directed toward the centre O of ω. The case that X is interior is typical enough:



   Thus m meets ω at A', B', and if R is the radius,

$$\begin{aligned} XB' &= (XO + OB') = (+d + R) \\ XA' &= (XO + OA') = (+d - R). \end{aligned}$$

   Hence

$$\begin{aligned} XA \cdot XB &= XA' \cdot XB' \quad \text{(theorem 1 [\textbf{chords through int. pt.}] in § 3)} \\ &= (d + R) \cdot (d - R) \\ &= d^2 - R^2, \qquad \text{(the power of } X\text{).} \end{aligned}$$

5. **Exercise 10.** Verify the other two cases, when X is on ω, or outside ω.

6. **Exercise 11.** Suppose $k > -R^2$ is a constant. What is the locus of points with power k?

7. **Exercise 12.** Suppose PT and PU are tangents from P to two concentric circles, with T on the smaller; and suppose segment PT meets the larger circle at Q. Prove that

$$PT^2 - PU^2 = QT^2.$$

6 Coordinates for the power, the radical axis, etc.

1. **More Problems 13**: Suppose a circle ω has

$$\begin{aligned} \text{centre} = O &= (a, b) \\ \text{radius} &= R. \end{aligned}$$

   (a) What is the distance d from any point $X = (x, y)$ to O?

$$d = \underline{\hspace{6cm}}.$$

   (b) When exactly is X on the circle ω?

   (c) What is the equation for ω?

   (d) What is the power of any point X with respect to ω? Is this consistent with part (c)?

2. Notice that the circle ω with centre (a, b) and radius R has equation

$$(x-a)^2 + (y-b)^2 = R^2$$

or

$$x^2 - 2xa + a^2 + y^2 - 2yb + b^2 = R^2$$

or

$$x^2 + y^2 - 2xa - 2yb + c = 0,$$

where the constant

$$c = a^2 + b^2 - R^2.$$

We might represent another circle ω' by

$$x^2 + y^2 - 2xa' - 2yb' + c' = 0,$$

for constants a', b', c'.

Note, as well, that ω, ω' are **concentric** precisely when both $a = a', b = b'$.

3. **Exercise 14.** Compute the length of the tangent from $X = (-3, 5)$ to the circle with equation $x^2 + y^2 = y - x$.

4. **Theorem.** The locus of all points whose powers with respect to two **non-concentric** circles are equal is a line perpendicular to the line of centres of the two axes.

   Proof. By part (2), we may take the two circles to be

$$\begin{aligned} x^2 + y^2 - 2xa - 2yb + c &= 0 \\ x^2 + y^2 - 2xa' - 2yb' + c' &= 0 \end{aligned}$$

   for certain constants $a, b, c,\ a', b', c'$, with either $a \neq a'$ or $b \neq b'$ (or both). In fact, we could just as well choose our x, y axes so that the two centres lie on the $x-$axis, meaning $b = b' = 0$.

   So how do we describe the desired locus? (A **locus** is a curve defined by some algebraic condition on (x, y).)

   Thus our locus is indeed the line

$$x = \frac{c' - c}{2(a' - a)} \ ,$$

   which is a (vertical) line perpendicular to the (horizontal) line of centres.

5. **Definition.** The above line associated to the two non-concentric circles ω, ω' is called their **radical axis**.

6. **Exercise 15.**

   (a) What is the radical axis when our two circles intersect?

   (b) What happens when the two circles are tangent?

   (c) Where does the radical axis fit into our discussion fo the intersecting pencil $\mathcal{I}$ and non-intersecting pencil $\mathcal{N}$ in §?

   (d) What happens when we attempt to describe the radical axis for **concentric** circles?

7. **Exercise 16.** When the distance between the centres of two circles exceeds the sum of their radii, the two circles have four common tangents. Prove that the midpoints of thee four segments are collinear (i.e., lie on one line).

8. **Exercise 17.** Given two non-collinear circles ω, ω', describe a ruler and compasses construction for their radical axis. (Hint: draw any third circle which meets ω in two points, say A and B, and ω' in two further points A' and B'. Suppose lines AB, $A'B'$ meet at X. What can you say about the power of X with respect to each of the three circles?)

9. **Exercise 18.** Suppose ω, ω' are two non-intersecting circles. Describe a ruler and compasses construction for the limit points of the non-intersecting pencil which contains ω, ω'.

7 Some Common Theorems in Euclidean Geometry

Since mathematics, in particular geometry, grows step-by-step from simpler to more and more complicated things, it must be that[1]

'what you can assume in your proof depends on where you are! '

Generally, in problems on contests or from magazines, you are allowed to assume and use any of the geometrical results typically covered in high school courses of the past. Here is a list of such theorems which come to mind. The list isn't complete, and some of the abbreviations are my inventions; but you will get a sense of what is available.

1. Basic Congruence. When two triangles such as $\triangle ABC$ and $\triangle DEF$ are identical in all respects they are called **congruent**, written $\triangle ABC \equiv \triangle DEF$. It's important to list the vertices in corresponding order: A and D first, B and E second, C and F third. By comparing corresponding parts, one correctly concludes that $AB = DE$, $BC = EF$, $AC = DF$, $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.

We must assume as an axiom something like

[**s.a.s**]: If two triangles $\triangle ABC$ and $\triangle DEF$ have equal corresponding *sides* $AB = DE$, *included angles* $\angle B = \angle E$, and *sides* $BC = EF$, then (we conclude)

$$\triangle ABC \equiv \triangle DEF \ ,$$

i.e. $AC = DF$, $\angle A = \angle D$ and $\angle C = \angle F$.

From this we can prove

[**a.s.a.**]: involving two angles and included side

[**s.s.s**]: involving three sides (triangles 'rigid')

[**r.h.s**]: involving the hypotenuse and one other side for each of two right triangles

However, congruence is not implied by the ambiguous criterion [**s.s.a.**].

[**$=\angle$'s in isosc. $\triangle$**]

[**$\triangle$ inequ.**]: each side of a triangle is less than the sum of the other two.

2. Angles and Parallels

[**= alt. angles**]: lines are parallel if and only if they meet any transversal in equal alternate angles.

[**$\angle$ sum in $\triangle$**]

[**ext. $\angle$**]

[1] Regrettably, these days the high school curriculum has mostly abandoned geometry as a story built on the the telling of proofs. Instead, you often get a mishmash of things, typically returning to the same topic over and over. This is very boring for students who like math!

3. Similarity and Related Results

[= intercepts] If any number of parallel lines make equal intercepts with a transversal m, then they make equal intercepts with any other transversal n.

[ratio thm] If PQ is parallel to BC in $\triangle ABC$, with P on AB and Q on AC, then

$$\frac{AP}{PB} = \frac{AQ}{QC}$$

Recall $\triangle DEF$ is *similar* to $\triangle ABC$, written $\triangle DEF \sim \triangle ABC$, if

$$\frac{DE}{AB} = \frac{DF}{AC} = \frac{EF}{BC}\ .$$

That is, the three ratios of corresponding sides are equal.

[a.a.a]: triangles with equal corresponding angles are similar (but usually not congruent)

[pythag]: In a right triangle $\triangle ABC$, with sides a, b and hypotenuse c, we have $a^2 + b^2 = c^2$.

(The converse to Pythagoras is also true and is often used.)

[law of sines]

[law of cosines]

4. Circles

[$\angle$ at arc]

[$\angle$ in semi $\bigcirc$]

[$\angle$'s in cyc. quad]

[chords through int. pt.]

[sec. and tang. from ext. pt.]

8 Extra Problems.

1. $ABCD$ is a quadrilateral inscribed in a circle; AB and CD are each equal to the radius. AC and BD meet in E. Find the number of degrees in $\angle AEB$.

2. $ABCD$ is a quadrilateral inscribed in a circle and $AB = CD$. Prove that $AC = BD$.

3. ABC is an equilateral triangle inscribed in a circle; P is any point on the minor (i.e. smaller) arc AC; BD is cut off on BP equal to CP. Prove:

   (a) $\triangle ABD \equiv \triangle ACP$,

   (b) $\triangle ADP$ is equilateral,

   (c) $BP = AP + PC$.

   In classical constructions, you are allowed to use only compasses and a straight edge, i.e., one side of a ruler whose markings you must ignore.

   Typically you are given certain data to work with. For example, lengths are normally given by a line segment (which you can copy using the compasses).

4. Show how to construct a right triangle given just the hypotenuse and one side.

5. Two circles intersect at A and B. AP and AQ are diameters. Prove PBQ is a straight line.

6. AB is the diameter of a circle. With centre B and radius AB, a second circle is drawn. Prove that any chord of the second circle through A is bisected by the circumference of the first.

7. Show how the square corner of a sheet of paper may be used to locate a diameter of a circle whose centre is unknown.

8. A, B, C are three points on a circle such that ABC is an acute-angled triangle. BQ and CR are diameters of the circle and AQ and AR are joined. Prove that $\angle BAR = \angle CAQ$.

9. If a triangle be inscribed in a circle and an angle be taken in each of the arcsb outside the triangle, the sum of these angles is four rt. angles.

10. $ABCD$ is a quadrilateral inscribed in a circle and $AD||BC$ (parallel). Prove $\angle B = \angle C$ and $AC = BD$.

11. A quadrilateral is inscribed in a circle. Find the sum of the angles subtended in the four arcs outside the quadrilateral.

12. $KLMN$ is a parallelogram (= quadrilateral, with pairs of opposite sides parallel). A circle through K and L meets KN and LM, produced if necessary, in P and Q respectively. Prove P, Q, M, N concyclic.

13. The point P is external to the circle ω with centre O.

    (a) Show how to construct (with ruler and compasses) the circle whose diameter is OP.

    (b) Now show how to construct the tangent lines from P to ω.

    (c) Prove that these two tangents are equal in length, and make equal angles with the line OP.

14. A circle inscribed in $\triangle ABC$, touches (i.e., is tangent to) BC, CA, AB in X, Y, Z respectively. Prove that $AZ + BX + CY$ is equal to one-half the perimeter of $\triangle ABC$.

15. Two parallel tangents to a circle, meet a third tangent at P and Q. Prove that PQ subtends a right angle at the centre.

16. The sides of a quadrilateral $ABCD$ touch a circle. AD and BC are produced to meet in X, and BA and CD are produced to meet in Y. Prove that the difference between AX and CX is equal to the difference between AY and CY.

17. The sides $BC,\ CA,\ AB$ of $\triangle ABC$ touch a circle at $D,\ E,\ F$ respectively. If $\angle A = 50°,\ \angle B = 78°$, calculate each angle of $\triangle DEF$.

18. Two concentric circles are drawn with radii 13 cm. and 5 cm. Calculate the length of a chord of the larger circle which touches the smaller.

19. If a chord AB of a circle ABC is parallel to the tangent at C, prove that $AC = BC$.

20. AB is a chord of a circle and AC is a diameter. AD is drawn perpendicular to the tangent at B. Show that AB bisects $\angle DAC$.

21. Two circles intersect at A and B and one of them passes through O, the centre of the other. Prove that OA bisects the angle between the commond chord AB and the tangent to the first circle at A.

22. A chord CD in a circle, perpendicular to a diameter AB, meets AB in E. Prove that $AE \cdot EB = CE^2$.

23. Through any point P on the commond chord MN of two intersecting circles, lines $APB,\ CPD$ are drawn, one of them meeting the circumference of one circle in $A,\ B$, and the other meeting the circumference of the second circle in C and D. Prove that $PA \cdot PB = PC \cdot PD$.

24. PQ is the common chord of two intersecting circles. AB and CD are chords in the circles which both pass through O, any point in PQ. Prove that $A,\ B,\ C,\ D$ lie on a circle.

25. Through points P and Q on a circle, straight lines $APB,\ CQD$ are drawn meeting a concentric circle in $A,\ B$, and $C,\ D$ respectively. Prove that $AP \cdot PB = CQ \cdot QD$.

26. Two chords $AB,\ CD$ of a circle intersect at an internal point X. Prove

$$AB^2 + XC^2 + XD^2 = CD^2 + XA^2 + XB^2$$

.

27. If two circles intersect, the tangents drawn to them from any point on the common chord produced, are equal.

28. If two circles intersect, their common chord (suitably extended) bisects their common tangents.

29. A triangle ABC is inscribed in a circle. A straight line drawn parallel to the tangent at A meets $AB,\ AC$ at $D,\ E$ respectively. Prove that $AB \cdot AD = AC \cdot AE$.

30. The common chord of two intersecting circles is produced to a point A. From A, two lines are drawn, one to cut one of the circles at B and C and the other to cut the second circle at D and E. Show that $B,\ C,\ E$ and D are concyclic.

9 Axioms and Proofs

Often when students and other users are doing mathematics, they actually have some application in mind and are really engaged in calculations of one sort or another. From basic arithmetic to advanced calculus, users of mathematics are frequently engaged in doing computations, generally following some explicit procedure (called an *algorithm*).

We have done a fair number of calculations during the Math Camp. *There is, however, much more to mathematics than this.*

In fact, a mathematician may be far more concerned with understanding *why* things work, typically by thinking about the patterns that underlie our geometric and arithmetic calculations. This thinking generally culminates in a proof.

Why proof? – because in mathematics we want to work with powerful and wide-ranging statements and methods, and we want to be 100% certain that these statements are true.

We should realize that the word 'true' has a very strong meaning. For example, in our **Theorem** [2] that the sum of the angles in a triangle is $180°$, we are in fact asserting that the sum of the angles is exactly this (not $179.000000003°$); and furthermore this is so for all triangles, from microscopic to astronomical size.

Now, in order to **prove** such statements, we necessarily work with simpler statements and ideas. We have a problem! This process will never end, unless at some convenient point we begin with an unquestioned set of **axioms** (taken to be true, but unproved) and some set of **terms** (taken to meaningful, yet undefined). To see this process in action, let's recall what happened in our proof that the angles of a triangle add to $180°$.

First of all, we should begin with a clean mental state. Thus, for the moment, we suppose that we don't know one way or the other if $\angle A + \angle B + \angle C = 180°$. Of course, our mind isn't (or shouldn't) be completely empty.We *do* know some geometrical facts, perhaps having nothing to do with triangles. So we try to use these facts to explain our theorem. For example, we 'already knew' that

- there is 'a line m through vertex C which is parallel to the opposite side AB'.
- 'alternate angles are equal'.
- 'any straight angle at C has $180°$.

Hold on! How did we know these other assertions to be true? Well, they must themselves have been proved or somehow accepted earlier on. And in the course of these earlier proofs, we used facts encountered or proved earlier still, and so forth. *Where does this all end*?

There is also the issue of mathematical language. While we are pursuing all these proofs, we are using various mathematical words, as well as exploiting the connections between these words. For example, even to discuss the theorem, we must already know what a 'triangle' is, what a '$180°$ angle' is, what 'parallel lines' are, etc.

Consider the word 'triangle'. Sometime early on in your mathematical experience, you were told what a triangle is, perhaps using pictures, or straws on a table. A more sophisticated description

[2] Theorems go by other names, like 'proposition', 'corollory', 'claim' or 'lemma', more or less depending on the stature of the result in mathematical society.

might go like this: 'a triangle is a closed plane figure bounded by three straight line segments'. But how do we define the words 'figure', 'straight line segment', etc. These too require explanation. *Where does this all end*?

The answer can only be that we must accept certain *primitive words*, e.g. *point* or *line*, as being undefined. There simply is no alternative to starting somewhere like this. Likewise, we must accept certain facts, called *axioms*, concerning our mathematical objects, as being true but unproved. This process of mathematical proof has been summarized very nicely by **H. S. M. Coxeter** in his beautiful book *Introduction to Geometry* (Wiley, 1969):

> In the logical development of geometry (or calculus or other branches of mathematics), each *definition* of a *concept* involves other concepts or relations. Thus, the only way to avoid a vicious circle is to accept certain *primitive* concepts and relations as undefined. Likewise, the proof of each proposition (or theorem) uses other propositions; and hence to again avoid a vicious circle, we must accept certain primitive propositions – called *axioms* or *postulates* – as true but unproved.

In a nutshell, we take certain primitive ideas, which everyone will be willing to believe and use them in a logical way as building blocks for more and more complicated results. This is the essence of the *deductive* method in mathematics. The force of the method is that if you believe the axioms, which are usually 'obviously true', then you *must* believe the theorems which follow, no matter how outlandish.

It is a wonderful fact indeed that many surprising, even bizarre, theorems can be proved on the basis of a certain number of obvious assumptions. It is perhaps even stranger that all this proves to be so often useful in 'real world' applications.

Of course, our axioms must not contradict one another. Otherwise we should be able to prove nothing useful at all. And if there are fewer axioms than very many, we shall be better able to understand what makes our mathematics work.

Beyond this, however, just what we choose to be an axiom, and what we reject, is a matter of taste and judgement: What does common sense suggest? What does our intuition suggest? What is the most elegant way to begin our mathematical journey?