Invariance - if there is repetiton, look for what does not change [1]

— THE UNB MATHEMATICS PROBLEM GROUP —

1. Seven cups are turned upside-down on a table. By repeatedly turning any four at a time, can you turn all the cups right-side up? What if you turn three cups at a time?

2. Suppose the positive integer n is odd. Alice writes all the integers $1, 2, \ldots, 2n$ on the board. Next she erases any two numbers and writes in their place only one number – the absolute value of their difference. After repeating this, prove that an odd number remains at the end.

3. A circle is divided into 6 sectors and the numbers $1, 0, 1, 0, 0, 0$ are put in counterclockwise order into the sectors. You may increase two neighboring numbers each by 1. Is it possible to equalize all the numbers after a sequence of such steps?

4. In the parliament of Outtaway, each MP has at most three enemies. Prove that the house can be split into two parties, so that each MP has at most one enemy in his or her party.

5. Starting with $(x_0, y_0) = (3, 2)$, generate a sequence of points in the plane by the rules
$$x_0 = 3, \quad y_0 = 2$$
$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \frac{2x_n y_n}{x_n + y_n} \quad .$$
Find $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$.

6. Think of an ordinary 8×8 chessboard. As often as you wish, you may repaint (to the opposite colour) all the squares of any row or column. Can you end up with just one black square?

7. Each of the numbers 1 to 10^6 is repreatedly replaced by its digital sum until we reach a list of 10^6 one-digit numbers. Will there be more 1's or 2's in this list?

[1] This phrase and some of the problems are taken from A. Engel, *Problem-Solving Strategies*.

8. A drill sergeant arranges mn cadets into an $m \times n$ rectangle. (Here m, n are positive integers.) Of course, the cadets of different heights are all over the place, so the sergeant rearranges each *column* by height, with shortest in front, tallest at the back. Then the sergeant notices that the rows are still messed up, so he arranges each *row* by height, with shortest on the left, tallest on the right. Oh no! Now he must look at the columns again. What happened?

9. A convex polygon has $2m$ vertices. In its interior we choose a point P which lies on no diagonal of the polygon. Prove that P lies on an even number of triangles whose vertices are amongst the $2m$ vertices of the polygon.

10. Solve
$$(x^2 - 3x + 3)^2 - 3(x^2 - 3x + 3) + 3 = x \quad .$$

11. A dragon has 100 heads. A knight can cut off 15, 17, 21 or 5 heads, with one swipe of his sword. But then 24, 2, 14 or 17 new heads, respectively, grow from the dragon's neck. Can the knight ever remove all the heads?

12. In a certain game one scores either a or b points, where a and b are certain positive integers. Given that there are thirty-five non-attainable scores, one of them 58, what are a and b?

13. We remove the first (left-most) digit of the number 7^{1996} then add it to the remaining number. This is repeated till a number with 10 digits remains. Prove that this number has two equal digits.

14. Consider the infinite sequence of squares
$$1, 4, 9, 16, \ldots .$$
Is there an infinite arithmetic subsequence? (That is, can one select in order infinitely many of these squares so that the difference between each two consecutive terms is some constant?)