Notes on Polynomials – from Barry Monson, UNB

1. Here are some polynomials and their degrees:

polynomial	degree	note
$6x^4 - 8x^3 + 21x^2 + 7x - 2$	4	quartic
$-2x^3 + 0x^2 + \frac{3}{2}x + \sqrt{2}$	3	cubic
$-2x^3 + \frac{3}{2}x + \sqrt{2}$	3	the same – but the quadratic term is 'implicit'
$5x^2 + x + 8$	2	quadratic
$at^2 + bt + c$	2	the variable here is t
$3x - \frac{7}{5}$	?	linear
16.3	?	constant
π	?	constant
0	?	the identically 0 polynomial

2. Why so important?

   - Polynomials model many physical and geometrical processes, such as the conic sections. E.g. the parabola describes the path of a ball tossed in the air, whereas the planetary orbits are ellipses.
   - Polynomials are easy to compute - you need only add, subtract, multiply the variable x with various constants. Compare this with $5\sin(76°) - 15.64(10^{0.112})$. Sure, a calculator does this, but how does your calculator work?

     In fact, your calculator chip can be easily hard-wired to compute polynomials, which are then used to **approximate** useful functions like $\sin(x)$ or 10^x. Approximation does involve errors, but by design these are supposed to be beyond the limits of your calculators's display. This very important branch of *applied mathematics* is a major application of *calculus*.

3. Generally, any expression like

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a polynomial in the variable x. The constants $a_n, a_{n-1}, \ldots, a_1, a_0$ are the *coefficients* of the polynomial.

Quick Problems:

(a) Why does $a_1 x$ fit the pattern? Why does a_0 fit the pattern?

(b) Careful – the degree might not be n. In fact, what is necessary for the degree to be n?

4. From a different point of view, polynomials make ‘nice’ functions. As I wrote above, they are easy to compute. Also, their graphs don’t do ‘weird’ things.

5. **Exercises in computing**. Let

$$p(x) = 6x^2 + 2x + 9 \, .$$

   Compute

   (a) $p(0)$. Why is this easy?

   (b) $p(1)$. Why is this easy?

   (c) $p(-1)$. Why is this easy?

   (d) $p(-2)$.

   (e) $p(10)$. Why is this easy?

   (f) $p(100)$. Why is this easy?

   (g) $p(x+1)$.

   (h) $p(t)$.

   (i) $p(t^2)$.

   (j) $p(2x)$.

   (k) $2p(x)$.

   (l) $p(x) - p(x)$.

   (m) $p(p(x))$.

6. In fact, polynomials can be added, subtracted, multiplied in a way very like what we do for ordinary numbers. If we are careful about remainders, then polynomials can also be divided, much as we do long division of numbers.

   Example. Look at these two typical polynomials, *in the same variable* x:

$$p = 6x^2 + 2x + 9 \, ,$$

$$f = 18x^3 + 25x - 4 \, .$$

   We’ll compute $p + f$, $p - f$, pf, and attempt to divide p into f.

7. In fact, the division process works for any polynomials, whose coefficients are real or even complex numbers.

   Theorem on Polynomial Division If $f(x)$ is any polynomial and $p(x) \neq 0$ is any *non-zero* polynomial, then one can compute exactly one

   quotient polynomial $q(x)$, and

   remainder polynomial $r(x)$, with $\deg(r(x)) < \deg(p(x))$, where

   $$f(x) = p(x)q(x) + r(x) \quad .$$

   Note that like all theorems in mathematics, this result must be and can be proved. We will merely see by examples that it is quite reasonable.

8. Compare what happens for ordinary numbers, like $F = 291$, $P = 7$. Here we don't have degrees to work with. Instead, the remainder R must be non-negative yet *smaller than* the positive divisor P:

   $$0 \leq R < P \quad .$$

9. What do we say when the remainder $r(x) = 0$, the zero polynomial?

10. We'll look at the special situation when the divisor $p(x) = x - k = 1x - k$ is linear. Here, the leading (linear) coefficient is 1, and the constant term k is *subtracted*. This is no loss of generality, since, for example,

    $$x + 4 = x - (-4) \quad .$$

11. **Remainder Theorem** The polynomial $f(x)$ has the linear factor $x - k$ precisely when $f(k) = 0$.

12. Show that

    (a) $x - 2$ is a factor of $3x^2 - 2x - 8$.

    (b) $x - 2$ is not a factor of $3x^2 - 2x - 6$.

    (c) $x - 2$ is a factor of $5x^3 + x^2 - 30x + 16$.

    (d) $x + 1$ is a factor of $3x^2 - 2x - 5$.

    (e) $2x - 3$ is a factor of $2x^3 - 13x^2 + 29x - 21$.

13. Find b if $x - 3$ is a factor of $2x^3 + bx - 90$.

14. Factor as far as possible the polynomial

    $$2x^3 - 13x^2 + 29x - 21 \quad .$$

    How does your answer change if you restrict yourself to polynomials whose coefficients are integers? rational numbers? real numbers? complex numbers?

15. **Philosophical Aside:** In fact, it seems from mathematical history that 'new kinds of numbers' have been discovered mainly in the attempt to solve more kinds of polynomial equations. Thus in ancient cultures the counting numbers $1, 2, 3, \ldots$ caused little controversy, though they were not written using modern decimal notation. As mathematical theory and applications expanded, it was necessary to 'invent'

    - rational numbers like $3/2$, which solves $2x - 3 = 0$;
    - irrational numbers, like $\sqrt{5}$, which solves $x^3 + x^2 - 5x - 5 = 0$;
    - complex numbers like i, which solves $x^2 + 1 = 0$.

16. About 200 years ago, the great German mathematician **Gauss** proved that complex numbers, which had been discovered and were widely used by then, were essentially the final number system we needed. What he showed was that every polynomial equation (of degree at least 1) has a complex root. The proof is extremely difficult, and can really only be understood after two or three years of honours level math. classes at university.

17. So we won't prove it! However, we might as well use the result. Using the Remainder Theorem we can write Gauss' theorem as

    The Fundamental Theorem of Algebra. Over the complex numbers (which include all integers, rationals , real numbers) any polynomial

    $$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \quad ,$$

    of degree $n \geq 1$, completely factors as a product of n linear factors:

    $$f(x) = (x - k_1)(x - k_2) \cdots (x - k_n) \quad .$$

18. *Important Notes* (a) The leading coefficient of $f(x)$ is 1. This is a convenience, since other polynomials are easily reduced to this case:

    $$-3x^2 + 6x + 24 = (-3)(x^2 - 2x - 8) = (-3)(x - 4)(x + 2) \ .$$

    (c) Thus the equation

    $$f(x) = 0$$

    has n complex roots $k_1, k_2, \ldots, k_n$, **allowing repeats**. For example,

    $$4x^3 + 16x^2 + 5x - 25 = 0$$

    has the roots $-\frac{5}{2}, -\frac{5}{2}$ and 1, since

    $$4x^3 + 16x^2 + 5x - 25 = (2x + 5)(2x + 5)(x - 1) \ .$$

    The actual number of different roots is a tricky issue, as are counting just real roots, or rational roots.

(c) Gauss' proof, and all other proofs , show that a factorization must *exist*. However, no mechanism for actually finding the factors is possible in most cases.

Yes, there is a quadratic formula. Also, there are complicated procedures for dealing with equations of degree 3 or 4. However, soon after Gauss, the Norwegian mathematician Abel showed that one 'cannot' find the factors of certain fifth-degree polynomials in any exact way, even though the factors are there!

This was very disturbing philosophically. By the way, there are related issues concerning the factoring of whole numbers, which have applications in the security of modern data transmission networks (e.g. theft of credit card info. on the internet).

19. Actually, if a polynomial equation has **integer or rational coefficients**, there is a procedure for finding all **rational roots**. This procedure does require that we factor integers; however, even for computers, that becomes a problem only when the integers have say 100 digits or more.

    The Rational Root Theorem Suppose that the polynomial equation

    $$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0$$

    has *integer coefficients only.*

    If the rational number $\frac{p}{q}$ is a root of the equation and is in lowest terms, **then**

    - the denominator q is a divisor of the leading coefficient a_n,
    - the numerator p is a divisor of the constant coefficient a_0.

20. **Note**: We can always assume the leading term $a_n \neq 0$. In fact, after tallying zero roots, we can also assume $a_0 \neq 0$. Then there are only finitely many possibilities for p, q and $\frac{p}{q}$. This will become clear in examples.

21. **Problems:**

    (a) Solve $5x^3 - 8x^2 - 2x + 3 = 0$.

    (b) Solve $-10x^5 + 16x^4 + 4x^3 - 6x^2 = 0$.

Problems concerning Polynomials

Many of these problems have appeared on Descartes exams, the Canadian Open Mathematics Challenge, the magazine *Quantum*, etc.

1. If $x^2 - x - k = 0$ and $3x^2 + kx + m = 0$ have the same roots, determine $k + m$.
2. Find k, m, t if
$$3x^3 + kx^2 - 2mx + t = 0$$
has roots $2, 3, -5$.
3. For what integers n is $n^2 - 4n - 21$ a prime number?
4. If a and b are roots of $3x^2 - 5x + 1 = 0$, find the equation whose roots are $\frac{a}{b}$ and $\frac{b}{a}$.
5. For which k does $x^3 + 4x^2 + 5kx - 2$ have the same remainder when divided by $x - 1$ or $x + 2$?
6. Solve simulataneously for x and y if
$$x^2 + y^2 = 5$$
$$xy = 2 \quad .$$
7. Solve simulataneously for x and y if
$$3x^2 - 2xy - y^2 = -33$$
$$x - y = 3 \quad .$$
8. Find x if
$$\frac{10^x}{10^x - 1} - \frac{1}{10^x} = \frac{3}{2} \quad .$$
9. If α and β are roots of the equation $x^2 - 3x - 5 = 0$, find the numerical value of $\alpha^2 + 2\alpha\beta + \beta^2$.
10. If α and β are roots of the equation $x^2 - 3x - 5 = 0$, find the numerical value of $\alpha^2 + \beta^2$.
11. If α and β are roots of the equation $x^2 - px + q = 0$, find the numerical value of $\alpha^2 + \beta^2$.
12. If α and β are roots of the equation $x^2 - px + q = 0$, find the numerical value of $1/\alpha + 1/\beta$.
13. If α and β are roots of the equation $x^2 - px + q = 0$, find the numerical value of $\alpha^3 + \beta^3$.
14. If α and β are roots of the equation $x^2 - px + q = 0$, find the numerical value of
$$\frac{\alpha + 2\beta}{\alpha - \beta} + \frac{2\alpha + \beta}{\beta - \alpha}.$$

15. If α, β, and γ are the roots of the equation

$$x^3 - px^2 + qx - r = 0,$$

express in terms of p, q, and r the values of:

(a) $\alpha^2 + \beta^2 + \gamma^2$;

(b) $\alpha^3 + \beta^3 + \gamma^3$;

(c) $\alpha^4 + \beta^4 + \gamma^4$;

(d) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$;

(e) $\dfrac{\alpha\beta}{\gamma} + \dfrac{\alpha\gamma}{\beta} + \dfrac{\beta\gamma}{\alpha}$.

16. Solve the following systems:

(a) $\begin{cases} \alpha + \beta &= 5 \\ \alpha\beta &= 6 \end{cases}$.

(b) $\begin{cases} \alpha + \beta &= 8 \\ \alpha\beta &= -7 \end{cases}$.

(c) $\begin{cases} \alpha + \beta + \gamma &= -9 \\ \alpha\beta + \alpha\gamma + \beta\gamma &= 19 \\ \alpha\beta\gamma &= -11 \end{cases}$.

(d) $\begin{cases} \alpha + \beta + \gamma &= 5 \\ \alpha^2 + \beta^2 + \gamma^2 &= 29 \\ \alpha\beta\gamma &= -24 \end{cases}$.

17. Let a, b, c, d, and e be integers such that both $a+b+c+d+e$ and $a^2+b^2+c^2+d^2+e^2$ are divisible by an odd integer n. Prove that $a^5 + b^5 + c^5 + d^5 + e^5 - 5abcde$ is also divisible by n.

18. Let r_1 and r_2 be roots of the equation $x^2 + 2x + 3 = 0$. Compute

$$\frac{r_1^2 + 4r_1 + 5}{r_1^2 + 5r_1 + 4} + \frac{r_2^2 + 4r_2 + 5}{r_2^2 + 5r_2 + 4}.$$

19. Find all triples (x, y, z) of real numbers such that

$$\begin{array}{rcl} x^2 &=& 4(y-1) \\ y^2 &=& 4(z-1) \\ z^2 &=& 4(x-1) \end{array}.$$

20. Factor $x^4 + 4$ over the integers.

21. Factor $4a^4 + b^4$ (over the integers).

22. Show that the number $4^n + n^4$ is composite for all integers $n > 1$.