The Theory Behind Stabilizer Chains

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First some definitions: if G is a group of permutations of a set Ω, a *base* for G is a list of elements $\omega_1, \omega_2, \ldots, \omega_s$ of Ω so that the stabilizer $G_{\omega_1,\omega_2,\ldots,\omega_s}$ equals 1. Here $G_{\omega_1,\omega_2,\ldots,\omega_r}$ is the stabilizer inside the subgroup $G_{\omega_1,\omega_2,\ldots,\omega_{r-1}}$ of ω_r, for each r. Let us write G_r instead of $G_{\omega_1,\omega_2,\ldots,\omega_r}$ and $G_0 = G$. In this situation the chain of subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_s = 1$$

is called a *stabilizer chain* (for G, with respect to the given base). We will consider for each r the subset Δ_r of Ω which is defined to be the G_r-orbit containing ω_{r+1}. Thus $\Delta_0 = \omega_1 G$, $\Delta_1 = \omega_2 G_1$ etc. A *strong generating set* for G (with respect to the base) is a set of generators for G which includes generators for each of the subgroups G_r. Thus in a strong generating set, G_r is generated by those generators which happen to fix each of $\omega_1, \ldots, \omega_r$.

PROPOSITION. *Each Δ_i is acted on transitively by G_i. As G_i-sets, $\Delta_i \cong G_{i+1}\backslash G_i$.*

Proof. We have $\omega_{i+1} \in \Delta_i$ and $\mathrm{Stab}_{G_i}(\omega_{i+1}) = G_{i+1}$. □

COROLLARY. $|G| = |\Delta_0| \cdots |\Delta_{s-1}|$.

Let H be a subgroup of a group G. A *right transversal* to H in G is the same thing as a set of right coset representatives for H in G, that is a set of elements $g_1, \ldots, g_t$ of G so that $G = Hg_1 \cup \cdots \cup Hg_t$.

PROPOSITION. *Let G act transitively on a set Δ and let $\omega \in \Delta$ be an element with stabilizer G_ω. Then elements $g_1, \ldots, g_t$ of G form a right transversal to G_ω in G if and only if $\Delta = \{\omega g_1, \ldots, \omega g_t\}$ and $t = |\Delta|$.*

Proof. This comes from the isomorphism of G-sets $\Delta \cong G_\omega \backslash G$ under which $\omega g \leftrightarrow G_\omega g$. □

This observation provides a way to compute a right transversal for G_{i+1} in G_i, for each i. It also suggests an algorithm to test whether a given permutation π of Ω is an element of G. We compute $(\omega_1)\pi$. If $\pi \in G$ this must equal $(\omega_1)g$ for some unique g in a right transversal for G_1 in G_0 and so $\pi g^{-1} \in G_1$. In fact, $\pi \in G$ if and only if $(\omega_1)\pi = (\omega_1)g$ for some g in the transversal and $\pi g^{-1} \in G_1$. We now continue to test whether $\pi g^{-1} \in G_1$ by repeating the algorithm.

How to compute a transversal to $\mathrm{Stab}_G(\omega)$? Take the generators of G and repeatedly apply them to ω, obtaining various elements of the form $\omega g_{i_1} g_{i_2} \cdots g_{i_r}$ where the g_{i_j} are generators of G. Each time we get an element we have seen previously, we discard it. Eventually we obtain the orbit ωG, and the various elements $g_{i_1} g_{i_2} \cdots g_{i_r}$ are a right transversal to $\mathrm{Stab}_G(\omega)$ in G.

In fact, what GAP does is to do the above with the inverses of the generators of G. If an inverse generator g^{-1} sends an already-computed element u to a new element v, the generator g is stored in position v in a list. This means that applying g to v gives u. By repeating this we eventually get back to the first element of the orbit. It is this list of generators that GAP stores in the field '`transversal`' of a stabilizer chain. Elements of a right transversal are obtained by multiplying the inverses of the generators in reverse sequence.

How does GAP compute generators for a stabilizer?

THEOREM (Schreier). *Let X be a set of generators for a group G, $H \leq G$ a subgroup, and T a right transversal for H in G such that the identity element of G represents the coset H. For each $g \in G$ let $\overline{g} \in T$ be such that $H\overline{g} = Hg$. Then*

$$\{tg(\overline{tg})^{-1} \mid t \in T, g \in X\}$$

is a set of generators for H.

Note that since $Htg = H\overline{tg}$, the elements $tg(\overline{tg})^{-1}$ lie in H always. Also $\overline{\overline{a}} = \overline{a}$, $\overline{\overline{a}b} = \overline{ab} = \overline{a\overline{b}}$. The generators in the set are called *Schreier generators*.

Proof. Suppose that $g_1 \cdots g_n \in H$ where the g_i lie in X. Then

$$g_1 \cdots g_n = (g_1 \overline{g_1}^{-1})(\overline{g_1} g_2 \overline{g_1 g_2}^{-1})(\overline{g_1 g_2} g_3 \overline{g_1 g_2 g_3}^{-1} \cdots (\overline{g_1 \cdots g_{n-1}} g_n)$$

is a product of the Schreier generators. Note that $g_1 \cdots g_n \in H$ so that $\overline{g_1 \cdots g_n} = 1$. □