Readings and Exercises on Groups

A. *The Theory of Groups*, J.J.Rotman

There are hundreds of suitable books. This one will do fine. Some, like this one, are 'right-to-left'; others, and this is my usual preference, are 'left-to-right'.

There are many exercises. Do write out solutions, or at least a brief summary with hints to yourself.

1. **Read all of Chapter 1.**

2. **Do Exercises**

   - 1.1, 1.2, 1.3, 1.4, 1.5
   - 1.6, 1.7, 1.8, 1.9, 1.10, 1.11
   - 1.12, 1.13, 1.14 [** need to know about equivalence relations **]
   - 1.15, 1.17, 1.18, 1.19, 1.20, 1.21, 1.22
   - 1.23, 1.24 [show false in non-commutative groups]
   - 1.25, 1.25

3. **Read all of Chapter 2.**

4. **Do Exercises**

   - 2.1
   - ** 2.6, 2.7, 2.9, 2.10, 2.11, 2.12
   - 2.13, 2.15, 2.20, 2.21, 2.22, 2.23, 2.24 [some number theory]
   - ** 2.25, 2.26, 2.27
   - 2.31, 2.32, 2.33, 2.34, 2.35, 2.36, 2.37
   - 2.42, 2.43, 2.44, 2.45, 2.53
   - Others of less importance: 2.3, 2.5, 2.14, 2.17, 2.19, 2.28, 2.38, 2.39

5. More exercises on multiplication of subsets of a group G. Suppose S, T, U are non-empty subsets of a group, perhaps not subgroups. The

operation is ordinary multiplication ab for $a, b \in G$. Recall that by definition

$$ST := \{st \,|\, s \in S, t \in T\} ,$$
$$S^{-1} := \{s^{-1} \,|\, s \in S\} .$$

One or the other of S and T could be singleton sets (one element). In such cases, we write sT instead of the more tiresome $\{s\}T$. As a typical example along this line, we have

$$aSb^{-1} = \{asb^{-1} \,|\, s \in S\} .$$

(Here a, b are fixed as s runs through S.)

(a) Prove $(ST)U = S(TU)$. (This is exercise 2.7.)

(b) Prove that S is a subgroup if and only if $SS = S$ and $S = S^{-1}$. Note: see Exercise 2.10; the second requirement is redundant when S is finite.

(c) If G is commutative, then $ST = TS$.

(d) For any non-commutative group G, give examples of subsets for which $ST = TS$. What about $ST \neq TS$?

****

By one of various equivalent definitions, a subgroup S of a group G is a **normal subgroup** of G if

$$aSa^{-1} \subseteq S \text{ for every } a \in G.$$

We indicate this special situation by writing

$$S \lhd G .$$

(e) **Theorem**. Prove that the following are equivalent for a subgroup S of the group G:

   i. S is normal in G.
   ii. $aSa^{-1} = S$ for every $a \in G$.
   iii. $aS = Sa$ for every $a \in G$.
   iv. Every right coset Sa is some left coset bS.
   v. Every left coset bS is some right coset Sa.
   vi. Every product of right cosets (i.e. $SaSb$) is some right coset.
   vii. Every product of left cosets is some left coset.

6. **Read most of chapter 3. If you wish, skip pp.42–46**

7. **Do Exercises**

   - 3.1, 3.2, 3.4, 3.5, 3.6
   - 3.8, 3.9, 3.11, 3.13, 3.14, 3.15, 3.16
   - 3.18, 3.20, 3.21, 3.25, 3.26
   - 3.36, 3.37, 3.38.